Uniqueness and stability properties of monostable pulsating fronts

François Hamel [a] and Lionel Roques [b] *

[a] Aix-Marseille Université, LATP, Faculté des Sciences et Techniques

Avenue Escadrille Normandie-Niemen, F-13397 Marseille Cedex 20, France

e-mail: francois.hamel@univ-cezanne.fr

& Helmholtz Zentrum München, Institut für Biomathematik und Biometrie

Ingolstädter Landstrasse 1, D-85764 Neuherberg, Germany

[b] INRA, UR546 Biostatistique et Processus Spatiaux, F-84914 Avignon, France

e-mail: lionel.roques@avignon.inra.fr

Abstract

In this paper, we prove the uniqueness, up to shifts, of pulsating traveling fronts for reaction-diffusion equations in periodic media with Kolmogorov-Petrovsky-Piskunov type nonlinearities. These results provide in particular a complete classification of all KPP pulsating fronts. Furthermore, in the more general case of monostable nonlinearities, we also derive several global stability properties and convergence to the pulsating fronts for the solutions of the Cauchy problem with front-like initial data. In particular, we prove the stability of KPP pulsating fronts with minimal speed, which is a new result even in the case when the medium is invariant in the direction of propagation.

1 Introduction and main results

This paper is the follow-up of the article [20] on qualitative properties of pulsating traveling fronts in periodic media with monostable reaction terms. By monostable we mean that the fronts connect one unstable limiting state to a weakly stable one. In [20] we proved monotonicity properties and exponential decay of these fronts. Here, we first show the uniqueness of KPP pulsating fronts, for any given speed. The second part of the paper is devoted to further stability properties for the solutions of the Cauchy problem with front-like initial data, for general monostable nonlinearities. All these issues have been left open so far

*The first author is indebted to the Alexander von Humboldt Foundation for its support. Both authors are also supported by the French "Agence Nationale de la Recherche" within projects ColonSGS, PREFERED, and URTICLIM (second author).

and the present paper fills in the main remaining gap in the theory of monostable and specific KPP traveling fronts in periodic media, in the sense that it provides a positive answer to the question of the classification and stability of all KPP pulsating fronts, as well as the stability of pulsating fronts with non-critical speeds in the general monostable framework. Lastly, we point out that, due to our general assumptions on the limiting stationary states, our stability results are new even in the special case of media which are invariant by translation in the direction of propagation. The stability of KPP fronts with minimal speeds involves completely new ideas and is an original result even in the most simplified situations which were previously considered in the literature.

1.1 General framework and assumptions

We consider reaction-diffusion-advection equations of the type

$$\begin{cases} u_t - \nabla\cdot(A(x,y)\nabla u) + q(x,y)\cdot\nabla u = f(x,y,u), & (x,y)\in\overline{\Omega}, \\ \nu A(x,y)\nabla u = 0, & (x,y)\in\partial\Omega, \end{cases} \tag{1.1}$$

in an unbounded domain $\Omega\subset\mathbb{R}^N$ which is assumed to be of class $C^{2,\alpha}$ (with $\alpha>0$), periodic in d directions and bounded in the remaining variables. That is, there are an integer $d\in\{1,\cdots,N\}$ and d positive real numbers $L_1,\ldots,L_d$ such that

$$\begin{cases} \exists\ R\geq 0, & \forall\ (x,y)\in\Omega, \quad |y|\leq R, \\ \forall\ k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}\times\{0\}^{N-d}, & \Omega=\Omega+k, \end{cases}$$

where

$$x=(x_1,\cdots,x_d),\quad y=(x_{d+1},\cdots,x_N),\quad z=(x,y)$$

and $|\cdot|$ denotes the euclidean norm. Admissible domains are the whole space $\mathbb{R}^N$, the whole space with periodic perforations, infinite cylinders with constant or periodically undulating sections, etc. We denote by ν the outward unit normal on $\partial\Omega$, and

$$\xi B\xi' = \sum_{1\leq i,j\leq N}\xi_i B_{ij}\xi'_j$$

for any two vectors $\xi=(\xi_i)_{1\leq i\leq N}$ and $\xi'=(\xi'_i)_{1\leq i\leq N}$ in $\mathbb{R}^N$ and any $N\times N$ matrix $B=(B_{ij})_{1\leq i,j\leq N}$ with real entries. Throughout the paper, call

$$C=\{(x,y)\in\overline{\Omega},\ x\in[0,L_1]\times\cdots\times[0,L_d]\}$$

the cell of periodicity of $\overline{\Omega}$.

Equations of the type (1.1) arise especially in combustion, population dynamics and ecological models (see e.g. [3, 16, 29, 35, 43, 49]), where u typically stands for the temperature or the concentration of a species.

The symmetric matrix field $A(x,y)=(A_{ij}(x,y))_{1\leq i,j\leq N}$ is of class $C^{1,\alpha}(\overline{\Omega})$ and uniformly positive definite. The vector field $q(x,y)=(q_i(x,y))_{1\leq i\leq N}$ is of class $C^{0,\alpha}(\overline{\Omega})$. The nonlinearity (x,y,u) $(\in\overline{\Omega}\times\mathbb{R})\ \mapsto f(x,y,u)$ is continuous, of class $C^{0,\alpha}$ with respect to (x,y) locally uniformly in $u \in \mathbb{R}$, and of class C^1 with respect to u. All functions A_{ij}, q_i and $f(\cdot,\cdot,u)$ (for all $u \in \mathbb{R}$) are assumed to be periodic, in the sense that they all satisfy

$$w(x+k,y) = w(x,y) \ \text{ for all } (x,y) \in \overline{\Omega} \text{ and } k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}.$$

We are given two $C^{2,\alpha}(\overline{\Omega})$ periodic solutions $p^\pm$ of the stationary equation

$$\begin{cases} -\nabla \cdot (A(x,y)\nabla p^\pm) + q(x,y) \cdot \nabla p^\pm = f(x,y,p^\pm) \ \text{ in } \overline{\Omega}, \\ \nu A(x,y)\nabla p^\pm = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

which are ordered, in the sense that $p^-(x,y) < p^+(x,y)$ for all $(x,y) \in \overline{\Omega}$.[1] We assume that there are $\beta > 0$ and $\gamma > 0$ such that the function

$$(x,y,s) \mapsto \frac{\partial f}{\partial u}(x,y,p^-(x,y)+s)$$

is of class $C^{0,\beta}(\overline{\Omega} \times [0,\gamma])$. Denote

$$\zeta^-(x,y) = \frac{\partial f}{\partial u}(x,y,p^-(x,y)). \tag{1.3}$$

Throughout the paper, we assume that p^- is linearly unstable in the sense that

$$\mu^- < 0, \tag{1.4}$$

where μ^- denotes the principal eigenvalue of the linearized operator around p^-

$$\psi \mapsto -\nabla \cdot (A(x,y)\nabla\psi) + q(x,y) \cdot \nabla\psi - \zeta^-(x,y)\ \psi$$

with periodicity conditions in $\overline{\Omega}$ and Neumann boundary condition $\nu A \nabla \psi = 0$ on $\partial\Omega$. That is, there exists a positive periodic function φ in $\overline{\Omega}$ such that

$$-\nabla \cdot (A(x,y)\nabla\varphi) + q(x,y) \cdot \nabla\varphi - \zeta^-(x,y)\varphi = \mu^-\varphi \ \text{ in } \overline{\Omega}$$

and $\nu A(x,y)\nabla\varphi = 0$ on $\partial\Omega$. Notice that the condition $\mu^- < 0$ is fulfilled in particular if $\zeta^-(x,y) > 0$ for all $(x,y) \in \overline{\Omega}$ or even if ζ^- is nonnegative and not identically equal to 0 in $\overline{\Omega}$. We also assume that there is ρ such that

$$0 < \rho < \min_{\overline{\Omega}} \, (p^+ - p^-)$$

and that, for any classical bounded super-solution $\overline{u}$ of

$$\begin{cases} \overline{u}_t - \nabla \cdot (A(x,y)\nabla\overline{u}) + q(x,y) \cdot \nabla\overline{u} \geq f(x,y,\overline{u}) \ \text{ in } \mathbb{R} \times \overline{\Omega}, \\ \nu A \nabla\overline{u} \geq 0 \ \text{ on } \mathbb{R} \times \partial\Omega, \end{cases}$$

[1] The present paper is concerned with uniqueness and stability properties of pulsating fronts connecting p^- and p^+. Under the assumptions below, the fact that these two limiting stationary states are ordered makes the fronts monotone in time, which plays an important role in the proofs.

satisfying $\overline{u} < p^+$ and $\Omega_{\overline{u}} = \{(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ \overline{u}(t,x,y) > p^+(x,y) - \rho\} \neq \emptyset$, there exists a family of functions $(\rho_\tau)_{\tau\in[0,1]}$ defined in $\overline{\Omega_{\overline{u}}}$ and satisfying

$$\begin{cases} \tau \mapsto \rho_\tau \text{ is continuous in } C^{1+\alpha/2;2+\alpha}_{t;(x,y)}\left(\overline{\Omega_{\overline{u}}}\right), \\ \tau \mapsto \rho_\tau(t,x,y) \text{ is non-decreasing for each } (t,x,y) \in \overline{\Omega_{\overline{u}}}, \\ \rho_0 = 0,\ \rho_1 \geq \rho,\ \inf\limits_{\overline{\Omega_{\overline{u}}}} \rho_\tau > 0 \text{ for each } \tau \in (0,1], \\ (\overline{u}+\rho_\tau)_t - \nabla\cdot(A\nabla(\overline{u}+\rho_\tau)) + q\cdot\nabla(\overline{u}+\rho_\tau) \geq f(x,y,\overline{u}+\rho_\tau) \ \text{ in } \Omega_{\overline{u},\tau}, \\ \nu A\nabla(\overline{u}+\rho_\tau) \geq 0 \ \text{ on } (\mathbb{R}\times\partial\Omega)\cap\Omega_{\overline{u},\tau}, \end{cases} \tag{1.5}$$

where

$$\Omega_{\overline{u},\tau} = \{(t,x,y) \in \Omega_{\overline{u}},\ \overline{u}(t,x,y) + \rho_\tau(t,x,y) < p^+(x,y)\}.$$

This condition is a weak stability condition for p^+. It is satisfied in particular if p^+ is linearly stable (as in Theorem 1.3 below), or if f is non-increasing in a left neighborhood of p^+, namely if there exists $\rho \in (0, \min_{\overline{\Omega}}(p^+ - p^-))$ such that $f(x,y,p^+(x,y)+\cdot)$ is non-increasing in $[-\rho,0]$ for all $(x,y) \in \overline{\Omega}$. It is straightforward to check that condition (1.5) is fulfilled as well if, for every $(x,y) \in \overline{\Omega}$, the function

$$s \mapsto \frac{f(x,y,p^-(x,y)+s) - f(x,y,p^-(x,y))}{s}$$

is non-increasing in $(0, p^+(x,y) - p^-(x,y))$. Indeed, in this case, we can take any ρ in $(0, \min_{\overline{\Omega}}(p^+ - p^-))$ (see Section 1.1 of [20] for details).

For some of our results, we shall assume a Kolmogorov-Petrovsky-Piskunov type condition on f, that is, for all $(x,y) \in \overline{\Omega}$ and $s \in [0, p^+(x,y) - p^-(x,y)]$,

$$f(x,y,p^-(x,y)+s) \ \leq \ f(x,y,p^-(x,y)) \ + \ \zeta^-(x,y)\ s. \tag{1.6}$$

As an example, when f depends on u only and admits two zeroes $p^- < p^+ \in \mathbb{R}$, the above conditions are satisfied if f is of class $C^{1,\beta}$ in a right neighborhood of p^- with $f'(p^-) > 0$ and if f is non-increasing in a left neighborhood of p^+. The KPP assumption (1.6) reads in this case:

$$f(u) \leq f'(p^-) \times (u - p^-) \ \text{ for all } u \in [p^-, p^+].$$

The nonlinearities $f(u) = u(1-u)$ or $f(u) = u(1-u)^m$ with $m \geq 1$ are archetype examples (with $p^- = 0$ and $p^+ = 1$) arising in biological models (see [16, 29]).

1.2 Uniqueness of KPP pulsating fronts

This paper is concerned with qualitative properties of an important class of solutions of (1.1), namely the pulsating traveling fronts connecting the two stationary states p^- and p^+. Given a unit vector $e \in \mathbb{R}^d \times \{0\}^{N-d}$, a pulsating front connecting p^- and p^+, traveling in the direction e with (mean) speed $c \in \mathbb{R}^*$, is a time-global classical solution $U(t,x,y)$ of (1.1) such that

$$\begin{cases} U(t,x,y) = \phi(x\cdot e - ct, x, y) \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega}, \\ (x,y)\mapsto \phi(s,x,y) \text{ is periodic in } \overline{\Omega} \text{ for all } s\in\mathbb{R}, \\ \phi(s,x,y) \underset{s\to\pm\infty}{\longrightarrow} p^{\mp}(x,y) \text{ uniformly in } (x,y)\in\overline{\Omega}, \\ p^-(x,y) < U(t,x,y) < p^+(x,y) \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega}. \end{cases} \tag{1.7}$$

With a slight abuse of notation, $x\cdot e$ denotes $x_1e_1+\cdots+x_de_d$, where $e_1,\ldots,e_d$ are the first d components of the vector e. The notion of pulsating traveling fronts extends that of usual traveling fronts which are invariant in the frame moving with speed c in the direction e. It was proved in [20] that any pulsating front is increasing in time if $c>0$ (or decreasing if $c<0$). More precisely, $\phi_s(s,x,y)<0$ for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$.[2]

Our first result is a uniqueness result, up to shifts in time, of the pulsating KPP traveling fronts for a given speed c in a given direction e.

Theorem 1.1 *Let e be a unit vector in $\mathbb{R}^d\times\{0\}^{N-d}$, let $c\in\mathbb{R}^*$ be given, and assume that the KPP assumption* (1.6) *is fulfilled. If $U_1(t,x,y)=\phi_1(x\cdot e-ct,x,y)$ and $U_2(t,x,y)=\phi_2(x\cdot e-ct,x,y)$ are two pulsating traveling fronts in the sense of* (1.7)*, then there exists $\sigma\in\mathbb{R}$ such that*

$$\phi_1(s,x,y)=\phi_2(s+\sigma,x,y) \quad \textit{for all } (s,x,y)\in\mathbb{R}\times\overline{\Omega}, \tag{1.8}$$

that is there exists $\tau\in\mathbb{R}$ ($\tau=-\sigma/c$) such that

$$U_1(t,x,y)=U_2(t+\tau,x,y) \textit{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega}. \tag{1.9}$$

As a consequence, in the KPP case, given any direction e and any speed $c\in\mathbb{R}^*$, the set of pulsating fronts $U(t,x,y)=\phi(x\cdot e-ct,x,y)$ is either empty or it is homeomorphic to $\mathbb{R}$. Notice indeed that if τ is not zero in (1.9), then $U_1\neq U_2$, since all fronts are strictly monotone in time (see [20]).

The existence of pulsating traveling fronts is known in some cases which are covered by the assumptions of Theorem 1.1. For instance, if

$$\begin{cases} p^-=0, \quad p^+=1, \quad f(x,y,u)>0 \text{ for all } (x,y,u)\in\overline{\Omega}\times(0,1), \\ f(x,y,u) \text{ is non-increasing with respect to } u \text{ in a left neighborhood of } 1, \\ \nabla\cdot q=0 \text{ in } \overline{\Omega},\ q\cdot\nu=0 \text{ on } \partial\Omega \text{ and } \displaystyle\int_C q_i(x,y)\,dx\,dy=0 \text{ for } 1\le i\le d, \end{cases} \tag{1.10}$$

if the KPP assumption (1.6) is satisfied, then, given any unit vector $e\in\mathbb{R}^d\times\{0\}^{N-d}$, there exists a minimal speed $c^*(e)>0$ such that pulsating traveling fronts exist if and only if

$$c\ge c^*(e)=\min_{\lambda>0}\left(-\frac{k(\lambda)}{\lambda}\right)=\min\,\{c\in\mathbb{R},\ \exists\ \lambda>0,\ k(\lambda)+\lambda c=0\}, \tag{1.11}$$

[2] In [20], the notation $U(t,x,y)=\phi(ct-x\cdot e,x,y)$ was used, with $\phi(\pm\infty,x,y)=p^{\pm}(x,y)$. In [20], ϕ was then increasing in s. The definition (1.7) makes U and ϕ face the same direction and is then more natural. In the present paper, the results of [20] have been translated in order to fit with the definition (1.7).

where $k(\lambda)$ is the principal eigenvalue of the operator

$$L_\lambda \psi := -\nabla \cdot (A\nabla\psi) + 2\lambda eA\nabla\psi + q\cdot\nabla\psi + [\lambda\nabla\cdot(Ae) - \lambda q\cdot e - \lambda^2 eAe - \zeta^-]\psi \qquad (1.12)$$

acting on the set of $C^2(\overline{\Omega})$ periodic functions ψ such that $\nu A\nabla\psi = \lambda(\nu Ae)\psi$ on $\partial\Omega$ (see [5], actually, this existence result has been proved under additional smoothness assumptions on the coefficients of (1.1)). Here $\zeta^-(x,y) = \frac{\partial f}{\partial u}(x,y,0)$. As already emphasized (see Section 1.1 in [20]), conditions (1.6) and (1.10) imply (1.4) and (1.5). Applications of the formula for the minimal speed $c^*(e)$ were given in [4, 13, 14, 24, 39, 41, 46, 50]. However, the uniqueness up to shifts for a given speed c was not known. Theorem 1.1 of the present paper then provides a complete classification of all pulsating fronts: namely, given a direction e in $\mathbb{R}^d \times \{0\}^{N-d}$, the set of pulsating fronts is a two-dimensional family, which can be parameterized by the speed c and the shift in the time variable.

For nonlinearities f satisfying (1.6) and (1.10), the derivative $\zeta^-(x,y) = \frac{\partial f}{\partial u}(x,y,0)$ is positive everywhere. This is why condition (1.4) is fulfilled automatically. However, if ζ^- is not everywhere positive, the principal eigenvalue μ^- may not be negative in general. In [7], nonlinearities $f = f(x,s)$ (for $x \in \Omega = \mathbb{R}^N$) satisfying

$$\begin{cases} p^- = 0, \quad f(x,0) = 0, \quad u \mapsto \dfrac{f(x,u)}{u} \text{ is decreasing in } u > 0, \\ \exists\, M > 0,\ \forall\, x \in \mathbb{R}^N,\ \forall\, u \geq M, \quad f(x,u) \leq 0 \end{cases} \qquad (1.13)$$

were considered, with no advection ($q = 0$). Typical examples are

$$f(x,u) = u(\zeta^-(x) - \eta(x)u),$$

where η is a periodic function which is bounded from above and below by two positive constants (see [43] for biological invasions models). Under the assumptions (1.13), the existence (and uniqueness) of a positive periodic steady state p^+ of (1.2) is equivalent to the condition

$$\mu^- < 0,$$

that is (1.4) (see [6]). Notice also that (1.13) implies (1.5) (see [20]), as well as (1.6). With the condition $\mu^- < 0$, the existence of pulsating fronts in any direction e was proved in [7] for all speeds $c \geq c^*(e)$, where $c^*(e)$ is still given by (1.11) (see also [25] for partial results in the one-dimensional case), and it was already known from [7] that no pulsating front exists with speed less than $c^*(e)$. However, the uniqueness of the fronts profiles in a given direction e and for a given speed $c \geq c^*(e)$ was still an open problem, even in dimension 1.

In short, the first part of the present paper gives a positive answer to the uniqueness issue of the KPP pulsating fronts, in a setting which unifies and is more general than (1.10) or (1.13). In particular, in this paper, the nonlinearity f is not assumed to be nonnegative or to satisfy any monotonicity properties. Actually, Theorem 1.1 follows from a more general uniqueness result which does not require the KPP assumption (1.6) but needs additional a priori properties for any two fronts with the same given speed, see Theorem 2.2 in Section 2.

Remark 1.2 If both p^- and p^+ are weakly stable –that is when (1.5) is satisfied and when the instability assumption (1.4) of p^- is replaced by a weak stability assumption which is similar to (1.5)–, then the analysis is much easier. Comparison principles such as Lemma 2.1 below, which can be viewed as weak maximum principles in some unbounded domains, would then hold not only in the region where the solutions are close to p^+, *but also* in the region where they are close to p^-. Two given fronts could then automatically be compared globally in $\mathbb{R} \times \overline{\Omega}$, up to time-shifts, and a sliding method similar to [2, 3] would imply that the functions $\phi(s, x, y)$ are unique up to shifts in the variable s, and that the speed c, if any, is necessarily unique. This is the case for instance for bistable or combustion-type nonlinearities (see [2, 3, 9, 32, 34, 47, 48, 49] for existence and further qualitative results with such reaction terms). In the present paper, as a consequence of the instability of p^-, one cannot use versions of the weak maximum principles in the region where the solutions are close to p^-. Therefore, even if the proofs of Theorems 1.1 and 2.2 below are based on a sliding method, the main difficulty is to compare two given fronts globally and especially to compare their tails in the region where they approach p^- (see Section 2 for further details).

1.3 Global stability of KPP or general monostable fronts

The second part of this paper is concerned with stability issues for KPP or general monostable fronts. The stability of the fronts and the convergence to them at large times is indeed one of the most important features of reaction-diffusion equations. We are back to the general periodic framework and we shall see that, under some assumptions on the initial conditions, the solutions of the Cauchy problem (1.1) will converge to pulsating fronts.

To state the stability results, we need a few more notations. In the sequel, e denotes a given unit vector in $\mathbb{R}^d \times \{0\}^{N-d}$ and $\zeta^-(x, y)$ is defined as in (1.3). For each $\lambda \in \mathbb{R}$, call $k(\lambda)$ the principal eigenvalue of the operator L_λ defined in (1.12) and let ψ_λ denote the unique positive principal eigenfunction of L_λ such that, say,

$$\|\psi_\lambda\|_{L^\infty(\Omega)} = 1. \tag{1.14}$$

It has been proved (see Proposition 1.2 in [20]) that, for any pulsating traveling front $U(t, x, y) = \phi(x \cdot e - ct, x, y)$ of (1.1) in the sense of (1.7), then

$$c \geq c^*(e) := \inf_{\lambda>0} \left(-\frac{k(\lambda)}{\lambda} \right). \tag{1.15}$$

The quantity $c^*(e)$ is a real number, and for each $c > c^*(e)$, the positive real number

$$\lambda_c = \min\{\lambda > 0,\ k(\lambda) + c\lambda = 0\} \tag{1.16}$$

is well-defined (see [20]).

Call now μ^+ the principal eigenvalue of the linearized operator

$$\psi \mapsto -\nabla \cdot (A(x, y)\nabla\psi) + q(x, y) \cdot \nabla\psi - \frac{\partial f}{\partial u}(x, y, p^+(x, y))\, \psi$$

around the limiting state p^+, with periodicity conditions in $\overline{\Omega}$ and Neumann boundary condition $\nu A \nabla \psi = 0$ on $\partial\Omega$. Let ψ^+ be the unique positive principal eigenfunction such that

$\|\psi^+\|_{L^\infty(\Omega)} = 1$. The function ψ^+ satisfies

$$\begin{cases} -\nabla\cdot(A(x,y)\nabla\psi^+) + q(x,y)\cdot\nabla\psi^+ - \dfrac{\partial f}{\partial u}(x,y,p^+(x,y))\,\psi^+ = \mu^+\psi^+ \ \text{ in } \overline{\Omega}, \\ \psi^+ > 0 \ \text{ in } \overline{\Omega}, \quad \max\limits_{\overline{\Omega}} \psi^+ = 1, \\ \nu A\nabla\psi^+ = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.17}$$

It is straightforward to check (see [20]) that the condition $\mu^+ > 0$ implies the weak stability property (1.5).

From now on, u_0 denotes a uniformly continuous function defined in $\overline{\Omega}$ such that

$$p^-(x,y) \le u_0(x,y) \le p^+(x,y) \ \text{ for all } (x,y) \in \overline{\Omega},$$

and let $u(t,x,y)$ be the solution of the Cauchy problem (1.1) for $t > 0$, with initial condition u_0 at time $t = 0$. Observe that

$$p^-(x,y) \le u(t,x,y) \le p^+(x,y)$$

for all $(x,y) \in \overline{\Omega}$ and $t \ge 0$, from the maximum principle.

The following theorem is concerned with the global stability of general monostable pulsating fronts for speeds larger than $c^*(e)$.

Theorem 1.3 *Assume that $\mu^+ > 0$ and that $U(t,x,y) = \phi(x\cdot e - ct, x, y)$ is a pulsating traveling front with speed $c > c^*(e)$, such that*

$$\lim_{s\to+\infty} \sup_{(x,y)\in\overline{\Omega}} \left| \frac{\ln(\phi(s,x,y) - p^-(x,y))}{s} + \lambda_c \right| = 0. \tag{1.18}$$

Then there exists $\varepsilon_0 > 0$ such that if

$$\liminf_{\varsigma\to-\infty} \inf_{(x,y)\in\overline{\Omega},\ x\cdot e\le\varsigma} \big[u_0(x,y) - p^+(x,y)\big] > -\varepsilon_0 \tag{1.19}$$

and

$$u_0(x,y) - p^-(x,y) \sim U(0,x,y) - p^-(x,y) \quad \text{as } x\cdot e \to +\infty,^{3} \tag{1.20}$$

then

$$\sup_{(x,y)\in\overline{\Omega}} |u(t,x,y) - U(t,x,y)| \to 0 \quad \text{as } t \to +\infty. \tag{1.21}$$

In Theorem 1.3, the assumption (1.18) on the logarithmic equivalent of $\phi(s,x,y) - p^-(x,y)$ as $s \to +\infty$ is automatically satisfied under the KPP condition (1.6), see formulas (1.22) and (1.23) below and Theorem 1.5. Actually, assumption (1.6) is not required here and it is only assumed that the limiting state p^- is unstable while the other one, p^+, is stable. But it does not mean a priori that f is of the KPP type or that there is no other stationary state

[3] Condition (1.20) is understood as $\sup_{(x,y)\in\overline{\Omega},\ x\cdot e\ge\varsigma} |(u_0(x,y) - p^-(x,y))/(U(0,x,y) - p^-(x,y)) - 1| \to 0$ as $\varsigma \to +\infty$.

p between p^- and p^+. In the general monostable case, assumption (1.18) is also fulfilled, without the KPP condition, as soon as there exists a pulsating front

$$U'(t,x,y) = \phi'(x\cdot e - c't, x, y)$$

in the sense of (1.7) with a speed $c' < c$, see Theorem 1.5 in [20]. As a consequence, the following corollary holds.

Corollary 1.4 *In Theorem* 1.3, *if the assumption* (1.18) *is replaced by the existence of a pulsating front* $U'(t,x,y) = \phi'(x\cdot e - c't, x, y)$ *with a speed* $c' < c$, *then the conclusion still holds.*

The existence of a pulsating front with a speed $c' < c$ is a reasonable assumption. For instance, under assumptions (1.10) with $\frac{\partial f}{\partial u}(x,y,0) > 0$, even without the KPP assumption (1.6), pulsating fronts $U(t,x,y) = \phi(x\cdot e - ct, x, y)$ exist if and only if $c \geq c^{**}(e)$, where the minimal speed $c^{**}(e)$ is such that $c^{**}(e) \geq c^*(e)$ and $c^*(e)$ is given in (1.11), see [2, 3]. Thus, for each $c > c^{**}(e)$, the existence of a pulsating traveling front with a speed $c' < c$ is guaranteed.

Let us now comment Theorem 1.3 and Corollary 1.4 and give some insight about their proofs. These two statements are global stability results for general monostable fronts. The initial condition u_0 is in some sense close to the pulsating front $U(0,\cdot,\cdot)$ at both ends, that is when $x\cdot e \to \pm\infty$. Assumption (1.19) means that u_0 has to be in the basin of attraction of the stable state p^+ as $x\cdot e$ is very negative. But these conditions are not very restrictive and u_0 is not required to be close to $U(0,\cdot,\cdot)$ when $|x\cdot e|$ is not large. Nevertheless, the convergence result (1.21) as $t \to +\infty$ is uniform in space. The only assumptions of Theorem 1.3 and Corollary 1.4 force the solution $u(t,x,y)$ to converge to the periodicity condition –namely the second property of (1.7)– asymptotically as $t \to +\infty$, whereas u_0 does not satisfy any such periodicity condition. A serious difficulty in Cauchy problems of the type (1.1) is indeed to get uniform estimates in the variables which are orthogonal to the direction e (establishing such estimates is an essential tool in the proof of Theorem 1.3). This difficulty was not present in the case of one-dimensional media or infinite cylinders with bounded sections, because of the compactness of the cross sections.

The general strategy of the proofs is, as in the paper by Fife and McLeod [15], to trap the solution $u(t,x,y)$ between suitable sub- and super-solutions which are close to some shifts of the pulsating traveling front U, and then to show that the shifts can be chosen as small as we want when $t \to +\infty$. However, the method is much more involved than in the bistable case investigated in [15]: not only the instability of p^- requires more precise estimates in the region where $x\cdot e - ct$ is positive, but the fact that p^+ is only assumed to be stable (in the sense that $\mu^+ > 0$) without any sign hypothesis for $f(\cdot,\cdot,s) - f(\cdot,\cdot,p^+)$ as $s \simeq p^+$ makes the situation more complicated and requires the use of the principal eigenfunction ψ^+ in the definition of the sub- and super-solutions (dealing here with the general monostable case introduces additional difficulties which would not be present in the KPP case, especially as far as the super-solutions are concerned). Furthermore, the dependence of all coefficients A, q and f on the spatial variables (x,y) induces additional technical difficulties, which are overcome by the use of space-dependent exponential correcting terms (we refer to Section 3 for further details).

Lastly, it is worth pointing out that there is no shift in the limiting profile, unlike for combustion-type or bistable equations (we refer to [15, 27, 40] for results with such nonlinearities in the one-dimensional case, or in infinite cylinders with invariance by translation in the direction of propagation, see equation (1.28) below).

Let us now deal with the particular KPP case (1.6). The assumptions of Theorem 1.3 can then be rewritten in a more explicit way. We first recall that, under the assumption (1.6), if $c > c^*(e)$ then

$$\phi(s,x,y) - p^-(x,y) \sim B_\phi\, e^{-\lambda_c s}\, \psi_{\lambda_c}(x,y) \text{ as } s \to +\infty \text{ uniformly in } (x,y) \in \overline{\Omega}. \tag{1.22}$$

for some $B_\phi > 0$, while if $c = c^*(e)$ then there is a unique $\lambda^* > 0$ such that $k(\lambda^*) + c^*(e)\lambda^* = 0$ and there exists $B_\phi > 0$ such that

$$\phi(s,x,y) - p^-(x,y) \sim B_\phi\, s^{2m+1}\, e^{-\lambda^* s}\, \psi_{\lambda^*}(x,y) \text{ as } s \to +\infty \text{ uniformly in } (x,y) \in \overline{\Omega}, \tag{1.23}$$

where $m \in \mathbb{N}$ and $2m+2$ is the multiplicity of λ^* as a root of $k(\lambda) + c^*(e)\lambda = 0$ (see Theorem 1.3 in [20]).

Theorem 1.5 *Assume that the KPP condition* (1.6) *is satisfied, that* $\mu^+ > 0$ *and that* $U(t,x,y) = \phi(x \cdot e - ct, x, y)$ *is a pulsating traveling front of* (1.1)*. Then there is* $\varepsilon_0 > 0$ *such that the following holds.*

1) *If* $c > c^*(e)$*, if* u_0 *fulfills* (1.19) *and if there exists* $B > 0$ *such that*

$$u_0(x,y) - p^-(x,y) \sim B\, e^{-\lambda_c x\cdot e}\, \psi_{\lambda_c}(x,y) \quad \text{as } x \cdot e \to +\infty, \tag{1.24}$$

then

$$\sup_{(x,y)\in\overline{\Omega}} |u(t,x,y) - U(t+\tau,x,y)| \to 0 \quad \text{as } t \to +\infty, \tag{1.25}$$

where τ *is the unique real number such that* $B_\phi\, e^{\lambda_c c \tau} = B$ *and* $B_\phi > 0$ *is given by* (1.22).

2) *If* $c = c^*(e)$*, if* u_0 *fulfills* (1.19) *and if there exists* $B > 0$ *such that*

$$u_0(x,y) - p^-(x,y) \sim B\,(x\cdot e)^{2m+1}\, e^{-\lambda^* x\cdot e}\, \psi_{\lambda^*}(x,y) \quad \text{as } x \cdot e \to +\infty, \tag{1.26}$$

then (1.25) *holds, where* τ *is the unique real number such that* $B_\phi\, e^{\lambda^* c^*(e)\tau} = B$ *and* $B_\phi > 0$ *is given by* (1.23).

It is immediate to see that, under the notations of Theorem 1.5, there holds

$$u_0(x,y) - p^-(x,y) \sim U(\tau,x,y) - p^-(x,y) \text{ as } x \cdot e \to +\infty.$$

As a consequence, part 1) of Theorem 1.5 is then a corollary of Theorem 1.3. Part 2) is more technical and needs a specific proof, which is done in Section 4. The main additional difficulty relies on the fact that the exponentially decaying functions $e^{-\lambda^* s}$ characterizing the behavior of the KPP fronts with minimal speeds near the unstable steady state p^- are multiplied by polynomial pre-factors. These pre-factors vanish somewhere. The construction of sub- and super-solutions must take this fact into account and it is therefore much more intricate. The sub- and super-solutions used in the proof use extra polynomial times exponentially decaying terms involving some derivatives of the principal eigenfunctions ψ_λ with respect to λ at the critical rate $\lambda = \lambda^*$. We point out that these ideas are new even in the previous special cases which were investigated in the literature.

From Corollary 1.4 and Theorem 1.5, it follows that the only case which is not covered by our stability results is the monostable case without the KPP assumption (1.6) and when the front U is the slowest one among all pulsating fronts. The situation is different in this case, and in general a shift in time is expected to occur in the convergence to the front at large times, like in combustion-type nonlinearities.

It can be seen from Theorems 1.3 and 1.5 that the propagation speed of $u(t,x,y)$ at large times strongly depends on the asymptotic behavior of the initial condition u_0 when it approaches the unstable state p^-. Actually, this fact had already been known in some simpler situations. In particular, the above stability results extend earlier ones for the usual traveling fronts $U(t,x) = \phi(x-ct)$ of the homogeneous one-dimensional equation

$$u_t = u_{xx} + f(u) \quad \text{in } \mathbb{R} \tag{1.27}$$

with $f(0) = f(1) = 0$ ($p^- = 0$ and $p^+ = 1$) and $f > 0$ in $(0,1)$, with or without the KPP condition $0 < f(s) \leq f'(0)s$ in $(0,1)$ (see e.g. [10, 18, 26, 30, 31, 42, 44, 45]). In this case, the minimal speed is equal to $c^* = 2\sqrt{f'(0)}$, $k(\lambda) = -\lambda^2 - f'(0)$ for each $\lambda \in \mathbb{R}$, $\lambda^* = \sqrt{f'(0)}$ and $m = 0$. Theorem 1.5 also generalizes the stability results for the traveling fronts $U(t,x,y) = \phi(x-ct,y)$ (which are still invariant in their moving frame) of equations of the type

$$u_t - \Delta u + \alpha(y)\,\frac{\partial u}{\partial x} = f(u),\ (x,y) \in \Omega = \mathbb{R} \times \omega, \quad \nu \cdot \nabla u = 0,\ (x,y) \in \partial\Omega \tag{1.28}$$

in straight infinite cylinders with smooth bounded sections ω and with underlying shear flows $q = (\alpha(y), 0, \ldots, 0)$, for nonlinearities f such that $f(0) = f(1) = 0$ and satisfying the stronger KPP assumption that $f(s)/s$ is non-increasing in $(0,1)$, see [33]. For equations (1.28), we refer to [9] for existence and uniqueness results of traveling fronts. Some stability results without the KPP assumption (when 0 and 1 are assumed to be the only possible steady states in $[0,1]$) have also been established in [33] and [40]. Recently, stability results for the one-dimensional equation

$$u_t = u_{xx} + f(x,u) \tag{1.29}$$

with KPP periodic nonlinearity $f(x,u)$ have been obtained in [1] with the use of Floquet exponents. The stability and uniqueness of one-dimensional pulsating KPP fronts for discretized equations have just been addressed in [19], under the assumption of exponential behavior of the fronts when they approach the unstable state. Actually, we point out that, in the KPP case, even for the equation (1.28) in infinite cylinders or for the one-dimensional periodic discrete or continuous framework, the question of the stability of the fronts with minimal speed was not known. Part 2) of Theorem 1.5 gives a positive answer to this important question.

The general philosophy of the aforementioned references [1, 33, 40] is that, if the initial condition u_0 approaches the unstable state $p^- = 0$ like a (pulsating) traveling front up to a faster exponential term, then the convergence of u to the front at large times is exponential in time in weighted functional spaces. The method is based on spectral properties in weighted spaces and it also uses the exact exponential behavior of the fronts when they approach 0. We conjecture that such a more precise convergence result holds in our general periodic framework –at least in the KPP case when the exact exponential behavior is known– under a stronger assumption on u_0, like $u_0(x,y)-p^-(x,y)=U(0,x,y)-p^-(x,y)+O((U(0,x,y)-p^-(x,y))^{1+\varepsilon})$ as $x\cdot e\to+\infty$, for some $\varepsilon>0$. However, this is not the purpose of the present paper and the method which we use to prove Theorems 1.3 and 1.5 is based directly on the construction of suitable sub- and super-solutions and on some Liouville type results. Furthermore, the method works in the general monostable periodic framework and it only requires that

$$u_0(x,y)-p^-(x,y)\sim U(0,x,y)-p^-(x,y)\ \text{ as } x\cdot e\to+\infty,$$

as well as the logarithmic equivalent of the fronts when they approach the unstable state p^-. However, in Theorems 1.3 and 1.5, the assumptions (1.20), (1.24) and (1.26) play an essential role and cannot be relaxed. Indeed, with KPP type nonlinearity f, for equation (1.29), if $u_0(x)$ is simply assumed to be trapped between two shifts of a front ϕ, then u may exhibit non-trivial dynamics and its ω-limit set may be a continuum of translates of ϕ, see [1]. On the other hand, even in the homogeneous one-dimensional case (1.27), if $u_0(x)$ is just assumed to be trapped as $x\cdot e\to+\infty$ between two exponentially decaying functions with two different decay rates, the asymptotic propagation speed of u as $t\to+\infty$ may not be unique in general, see [21] for details (see also [23] for results in the same spirit for combustion-type equations). Lastly, if $u_0(x)$ decays more slowly than any exponentially decaying function as $x\cdot e\to+\infty$, then the asymptotic propagation speed is infinite, see [11, 22].

1.4 Additional results in the time-periodic case

Finally, we mention that, with the same type of methods as in this paper, similar uniqueness and stability results can be established for pulsating fronts in time-periodic media (however, in order not to lengthen this paper, we just state the conclusions without the detailed proofs). Namely, consider reaction-diffusion-advection equations of the type

$$\begin{cases} u_t-\nabla\cdot(A(t,y)\nabla u)+q(t,y)\cdot\nabla u=f(t,y,u) & \text{in } \overline{\Omega},\\ \nu A\nabla u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.30}$$

in a smooth unbounded domain $\Omega=\{(x,y)\in\mathbb{R}^d\times\omega\}$, where ω is a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^{N-d}$. The uniformly elliptic symmetric matrix field $A(t,y)=(A_{ij}(t,y))_{1\le i,j\le N}$ is of class $C^{1,\alpha/2;1,\alpha}_{t;y}(\mathbb{R}\times\overline{\omega})$, the vector field $q(t,y)=(q_i(t,y))_{1\le i\le N}$ is of class $C^{0,\alpha/2;1,\alpha}_{t;y}(\mathbb{R}\times\overline{\omega})$ and the nonlinearity $(t,y,u)\ (\in\mathbb{R}\times\overline{\omega}\times\mathbb{R})\ \mapsto f(t,y,u)$ is continuous, of class $C^{0,\alpha/2;0,\alpha}$ with respect to (t,y) locally uniformly in $u\in\mathbb{R}$ and of class C^1 with respect to u in $\mathbb{R}\times\overline{\omega}\times\mathbb{R}$. All functions A_{ij}, q_i and $f(\cdot,\cdot,u)$ (for all $u\in\mathbb{R}$) are assumed to be time-periodic, in the sense that they satisfy $w(t+T,y)=w(t,y)$ for all $(t,y)\in\mathbb{R}\times\overline{\omega}$, where $T>0$ is given. We are given two time-periodic classical solutions $p^\pm$ of (1.30) satisfying

$$p^-(t,y)<p^+(t,y)\ \text{ for all } (t,y)\in\mathbb{R}\times\overline{\omega}.$$

Assume that the function $(t,y,s)\mapsto \frac{\partial f}{\partial u}(t,y,p^-(t,y)+s)$ is of class $C^{0,\beta}(\mathbb{R}\times\overline{\omega}\times[0,\gamma])$ for some $\beta>0$ and $\gamma>0$, and that $\mu^-<0$, where μ^- denotes the principal eigenvalue of the linearized operator around p^-

$$\psi(t,y)\mapsto \psi_t-\nabla\cdot(A(t,y)\nabla\psi)+q(t,y)\cdot\nabla\psi-\frac{\partial f}{\partial u}(t,y,p^-(t,y))\psi$$

with time-periodicity conditions in $\mathbb{R}\times\overline{\omega}$ and Neumann boundary condition $\nu A\nabla\psi=0$ on $\mathbb{R}\times\partial\omega$. With a slight abuse of notations, $\nabla\psi$ denotes $(0,\dots,0,\nabla_y\psi)\in\{0\}^d\times\mathbb{R}^{N-d}$. Assume that there is ρ such that $0<\rho<\min_{\mathbb{R}\times\overline{\omega}}(p^+-p^-)$ and, for any classical bounded super-solution $\overline{u}$ of (1.30) satisfying $\overline{u}<p^+$ and $\Omega_{\overline{u}}=\{\overline{u}(t,x,y)>p^+(t,y)-\rho\}\neq\emptyset$, there exists a family of functions $(\rho_\tau)_{\tau\in[0,1]}$ defined in $\overline{\Omega_{\overline{u}}}$ and satisfying (1.5) with $\Omega_{\overline{u},\tau}=\{(t,x,y)\in\Omega_{\overline{u}},\ \overline{u}(t,x,y)+\rho_\tau(t,x,y)<p^+(t,y)\}$. The KPP condition (1.6) is replaced with the following one: for all $(t,y)\in\mathbb{R}\times\overline{\omega}$ and $s\in[0,p^+(t,y)-p^-(t,y)]$,

$$f(t,y,p^-(t,y)+s)\ \le\ f(t,y,p^-(t,y))\ +\ \frac{\partial f}{\partial u}(t,y,p^-(t,y))\ s. \tag{1.31}$$

Given a unit vector $e\in\mathbb{R}^d\times\{0\}^{N-d}$, a pulsating front connecting p^- and p^+, traveling in the direction e with mean speed $c\in\mathbb{R}^*$, is a classical solution $U(t,x,y)$ of (1.30) such that

$$\begin{cases} U(t,x,y)=\phi(x\cdot e-ct,t,y)\ \text{ for all } (t,x,y)\in\mathbb{R}\times\mathbb{R}^d\times\overline{\omega},\\ \phi(s,t+T,y)=\phi(s,t,y) \text{ for all } (s,t,y)\in\mathbb{R}^2\times\overline{\omega},\\ \phi(s,t,y)\underset{s\to\pm\infty}{\longrightarrow}p^{\mp}(t,y) \text{ uniformly in } (t,y)\in\mathbb{R}\times\overline{\omega},\\ p^-(t,y)<U(t,x,y)<p^+(t,y) \text{ for all } (t,x,y)\in\mathbb{R}\times\mathbb{R}^d\times\overline{\omega}. \end{cases} \tag{1.32}$$

We refer to [17, 37, 38] for existence results and speed estimates of pulsating fronts for equations of the type (1.30) with time-periodic KPP nonlinearities and shear flows (see also [36] for the existence of fronts in space-time periodic media). For each $\lambda\in\mathbb{R}$, still define $k(\lambda)$ as the principal eigenvalue of the operator

$$\psi\mapsto\psi_t-\nabla\cdot(A\nabla\psi)+2\lambda eA\nabla\psi+q\cdot\nabla\psi+\left[\lambda\nabla\cdot(Ae)-\lambda q\cdot e-\lambda^2 eAe-\frac{\partial f}{\partial u}(t,y,p^-(t,y))\right]\psi$$

with time-periodicity conditions in $\mathbb{R}\times\overline{\omega}$ and boundary conditions $\nu A\nabla\psi=\lambda(\nu Ae)\psi$ on $\mathbb{R}\times\partial\omega$, and denote by ψ_λ the unique positive principal eigenfunction such that $\|\psi_\lambda\|_{L^\infty(\mathbb{R}\times\omega)}=1$. Define $c^*(e)$ as in (1.15) and for each $c>c^*(e)$, define $\lambda_c>0$ as in (1.16). These quantities are well-defined real numbers.

Then, for any pulsating traveling front, one has $c\ge c^*(e)$ (this fact had already been mentioned in [20]). Furthermore, under the KPP assumption (1.31), if

$$U_1(t,x,y)=\phi_1(x\cdot e-ct,t,y)\ \text{ and }\ U_2(t,x,y)=\phi_2(x\cdot e-ct,t,y)$$

are two pulsating travelling fronts with the same speed c, then $\phi_1(s,t,y)=\phi_2(s+\sigma,t,y)$ in $\mathbb{R}^2\times\overline{\omega}$ for some $\sigma\in\mathbb{R}$.

In the sequel, assume that $\mu^+ > 0$, where μ^+ denotes the principal eigenvalue of the linearized operator around p^+

$$\psi(t,y) \mapsto \psi_t - \nabla\cdot(A(t,y)\nabla\psi) + q(t,y)\cdot\nabla\psi - \frac{\partial f}{\partial u}(t,y,p^+(t,y))\,\psi$$

with time-periodicity in $\mathbb{R}\times\overline{\omega}$ and Neumann boundary condition $\nu A\nabla\psi = 0$ on $\mathbb{R}\times\partial\omega$. Consider a pulsating front $U(t,x,y) = \phi(x\cdot e - ct, t, y)$ in the sense of (1.32).

If $c > c^*(e)$ and if $\ln(\phi(s,t,y) - p^-(t,y)) \sim -\lambda_c s$ as $s \to +\infty$ uniformly in $(t,y)\in\mathbb{R}\times\overline{\omega}$, then there exists $\varepsilon_0 > 0$ such that, for any uniformly continuous function u_0 such that

$$\left\{\begin{array}{l} p^-(\tau,y) \le u_0(x,y) \le p^+(\tau,y) \ \text{ for all } (x,y)\in\overline{\Omega}, \\ \displaystyle\liminf_{\varsigma\to-\infty} \ \inf_{(x,y)\in\overline{\Omega},\ x\cdot e\le\varsigma} [u_0(x,y) - p^+(\tau,y)] > -\varepsilon_0 \end{array}\right. \tag{1.33}$$

and $u_0(x,y) - p^-(\tau,y) \sim U(\tau,x,y) - p^-(\tau,y)$ as $x\cdot e \to +\infty$ for some $\tau\in\mathbb{R}$, then the solution $u(t,x,y)$ of (1.30) with initial condition u_0 satisfies

$$\sup_{(x,y)\in\overline{\Omega}} |u(t,x,y) - U(t+\tau,x,y)| \to 0 \ \text{ as } t\to+\infty.$$

Lastly, under the KPP condition (1.31), there is $\varepsilon_0 > 0$ such that the following holds. If $c > c^*(e)$ and if there exist $\tau\in\mathbb{R}$ and $B > 0$ such that u_0 satisfies (1.33) and $u_0(x,y) - p^-(\tau,y) \sim B\,e^{-\lambda_c x\cdot e}\,\psi_{\lambda_c}(\tau,y)$ as $x\cdot e\to+\infty$, then the solution $u(t,x,y)$ of (1.30) with initial condition u_0 satisfies

$$\sup_{(x,y)\in\overline{\Omega}} |u(t,x,y) - U(t+\tau,x+\sigma e,y)| \to 0 \ \text{ as } t\to+\infty, \tag{1.34}$$

where σ is the unique real number such that $B_\phi\,e^{\lambda_c(c\tau-\sigma)} = B$ and $B_\phi > 0$ is given by:

$$\phi(s,t,y) - p^-(t,y) \sim B_\phi\,e^{-\lambda_c s}\,\psi_{\lambda_c}(t,y) \ \text{ as } s\to+\infty \ \text{ uniformly in } (t,y)\in\mathbb{R}\times\overline{\omega}.$$

On the other hand, if $c = c^*(e)$ and if there exist $\tau\in\mathbb{R}$ and $B > 0$ such that u_0 satisfies (1.33) and $u_0(x,y) - p^-(\tau,y) \sim B\,(x\cdot e)^{2m+1}\,e^{-\lambda^* x\cdot e}\,\psi_{\lambda^*}(\tau,y)$ as $x\cdot e\to+\infty$, where λ^* is the unique positive root of $k(\lambda) + c^*(e)\lambda = 0$, with multiplicity $2m+2$, then (1.34) holds, where σ satisfies $B_\phi\,e^{\lambda^*(c^*(e)\tau-\sigma)} = B$ and $B_\phi > 0$ is given by:

$$\phi(s,t,y) - p^-(t,y) \sim B_\phi\,s^{2m+1}\,e^{-\lambda^* s}\,\psi_{\lambda^*}(t,y) \ \text{ as } s\to+\infty, \text{ uniformly in } (t,y)\in\mathbb{R}\times\overline{\omega}.$$

Outline of the paper. Section 2 is devoted to the uniqueness results. In Section 3, the proof of the stability result in the general monostable case is done. Lastly, Section 4 is concerned with the proof of the stability of KPP fronts with minimal speed $c^*(e)$.

2 Uniqueness of the fronts up to shifts

This section is devoted to the proof of the uniqueness result, that is Theorem 1.1. Theorem 1.1 is itself based on another uniqueness result which is valid in the general monostable case. The basic strategy is to compare a given front ϕ_2 with respect to the shifts of another one ϕ_1 and to prove that, for a critical shift, the two fronts are identically equal. In other words, we use a sliding method. One of the difficulties is to initiate the sliding method, that is to compare the solutions globally in $\mathbb{R}\times\overline{\Omega}$, and in particular in the region where both fronts are close to p^- (as $s\to+\infty$). In this region, the weak maximum principle does not hold because of the instability of p^-. However, this difficulty can be overcome because the fronts have a nondegenerate behavior as $s\to+\infty$, see (2.2) below.

Before doing so, we first quote from [20] a useful lemma (see Lemma 2.3 in [20]) which is a comparison result between sub- and super-solutions in the region where $s\le h$.

Lemma 2.1 *Let $\rho\in(0,\min_{\overline{\Omega}}(p^+-p^-))$ be given as in (1.5). Let $\overline{U}$ and $\underline{U}$ be respectively classical super-solution and sub-solution of*

$$\begin{cases}\overline{U}_t-\nabla\cdot(A(x,y)\nabla\overline{U})+q(x,y)\cdot\nabla\overline{U}\ge f(x,y,\overline{U}) & \text{in } \mathbb{R}\times\overline{\Omega},\\ \nu A\nabla\overline{U}\ge 0 \quad \text{on } \mathbb{R}\times\partial\Omega,\end{cases}$$

and

$$\begin{cases}\underline{U}_t-\nabla\cdot(A(x,y)\nabla\underline{U})+q(x,y)\cdot\nabla\underline{U}\le f(x,y,\underline{U}) & \text{in } \mathbb{R}\times\overline{\Omega},\\ \nu A\nabla\underline{U}\le 0 \quad \text{on } \mathbb{R}\times\partial\Omega,\end{cases}$$

such that $\overline{U}<p^+$ and $\underline{U}<p^+$ in $\mathbb{R}\times\overline{\Omega}$. Assume that $\overline{U}(t,x,y)=\overline{\Phi}(x\cdot e-ct,x,y)$ and $\underline{U}(t,x,y)=\underline{\Phi}(x\cdot e-ct,x,y)$, where $\overline{\Phi}$ and $\underline{\Phi}$ are periodic in (x,y), $c\neq 0$ and $e\in\mathbb{R}^d\times\{0\}^{N-d}$ with $|e|=1$. If there exists $h\in\mathbb{R}$ such that

$$\begin{cases}\overline{\Phi}(s,x,y)>p^+(x,y)-\rho \quad \text{for all } s\le h \text{ and } (x,y)\in\overline{\Omega},\\ \overline{\Phi}(h,x,y)\ge\underline{\Phi}(h,x,y) \quad \text{for all } (x,y)\in\overline{\Omega},\\ \liminf\limits_{s\to-\infty}\left[\min\limits_{(x,y)\in\overline{\Omega}}(\overline{\Phi}(s,x,y)-\underline{\Phi}(s,x,y))\right]\ge 0,\end{cases}$$

then

$$\overline{\Phi}(s,x,y)\ge\underline{\Phi}(s,x,y) \quad \text{for all } s\le h \text{ and } (x,y)\in\overline{\Omega},$$

that is $\overline{U}(t,x,y)\ge\underline{U}(t,x,y)$ for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$ such that $x\cdot e-ct\le h$.

We then use the following general uniqueness result, which does not require the KPP assumption (1.6):

Theorem 2.2 *Let e be a unit vector in $\mathbb{R}^d\times\{0\}^{N-d}$ and $c\in\mathbb{R}^*$ be given. Assume that for any two pulsating traveling fronts $U(t,x,y)=\phi(x\cdot e-ct,x,y)$ and $U'(t,x,y)=\phi'(x\cdot e-ct,x,y)$ in the sense of (1.7), there exists a constant $C_{[\phi,\phi']}\in(0,+\infty)$ such that*

$$\frac{\phi(s,x,y)-p^-(x,y)}{\phi'(s,x,y)-p^-(x,y)}\to C_{[\phi,\phi']} \quad \text{as } s\to+\infty, \quad \text{uniformly in } (x,y)\in\overline{\Omega}. \tag{2.1}$$

Then, if $U_1(t,x,y)=\phi_1(x\cdot e-ct,x,y)$ and $U_2(t,x,y)=\phi_2(x\cdot e-ct,x,y)$ are two pulsating fronts, there exists $\sigma\in\mathbb{R}$ such that (1.8) *and* (1.9) *hold.*

Proof. Step 1. Let $U(t,x,y)=\phi(x\cdot e-ct,x,y)$ be any pulsating traveling front in the sense of (1.7). From Proposition 2.2 of [20], we know that there exist two positive real numbers $\lambda_{m,\phi}\le\lambda_{M,\phi}$ such that

$$\left\{\begin{array}{l}\lambda_{m,\phi}:=\liminf\limits_{s\to+\infty}\left(\min\limits_{(x,y)\in\overline{\Omega}}\dfrac{-\phi_s(s,x,y)}{\phi(s,x,y)-p^-(x,y)}\right)>0,\\ \lambda_{M,\phi}:=\limsup\limits_{s\to+\infty}\left(\max\limits_{(x,y)\in\overline{\Omega}}\dfrac{-\phi_s(s,x,y)}{\phi(s,x,y)-p^-(x,y)}\right)<+\infty.\end{array}\right.\tag{2.2}$$

For each $\sigma\in\mathbb{R}$, denote $C_{[\phi^\sigma,\phi]}$ the constant defined as in the statement of Theorem 2.2, with

$$\phi^\sigma(\cdot,\cdot,\cdot):=\phi(\cdot+\sigma,\cdot,\cdot).$$

Then, we claim that

$$\exists\ \nu>0,\quad\forall\ \sigma\in\mathbb{R},\quad C_{[\phi^\sigma,\phi]}=e^{-\nu\sigma}.\tag{2.3}$$

Indeed, for any $\sigma,\ \sigma'\in\mathbb{R}$ and $(x,y)\in\overline{\Omega}$,

$$\begin{array}{rcl}C_{[\phi^{\sigma+\sigma'},\phi]}&=&\lim\limits_{s\to+\infty}\dfrac{\phi(s+\sigma+\sigma',x,y)-p^-(x,y)}{\phi(s,x,y)-p^-(x,y)}\\ &=&\lim\limits_{s\to+\infty}\left(\dfrac{\phi(s+\sigma+\sigma',x,y)-p^-(x,y)}{\phi(s+\sigma',x,y)-p^-(x,y)}\times\dfrac{\phi(s+\sigma',x,y)-p^-(x,y)}{\phi(s,x,y)-p^-(x,y)}\right)\\ &=&C_{[\phi^\sigma,\phi]}\times C_{[\phi^{\sigma'},\phi]}.\end{array}$$

Furthermore, the function $\sigma\mapsto C_{[\phi^\sigma,\phi]}$ is non-increasing in $\mathbb{R}$ since $\phi(s,x,y)$ is decreasing in s (see Proposition 2.5 in [20]). As a consequence, there exists $\nu\in[0,+\infty)$ such that $C_{[\phi^\sigma,\phi]}=e^{-\nu\sigma}$ for all $\sigma\in\mathbb{R}$. Using (2.2), we finally obtain that $\nu\in[\lambda_{m,\phi},\lambda_{M,\phi}]$, whence $\nu>0$. This shows (2.3).

Step 2. Now, let $U_1(t,x,y)=\phi_1(x\cdot e-ct,x,y)$ and $U_2(t,x,y)=\phi_2(x\cdot e-ct,x,y)$ be two pulsating fronts satisfying (1.7). From (2.3) applied with $\phi=\phi_1$, we know that, for $\sigma<0$ negative enough,

$$C_{[\phi_1^\sigma,\phi_2]}=C_{[\phi_1^\sigma,\phi_1]}\times C_{[\phi_1,\phi_2]}>1.$$

Since ϕ_1 is strictly decreasing with respect to s, we deduce that there exist $\Sigma_0>0$, $\sigma_0<0$ such that

$$\forall\ \sigma\le\sigma_0,\quad\phi_2\le\phi_1^\sigma\ \text{ in }[\Sigma_0,+\infty)\times\overline{\Omega}.\tag{2.4}$$

Since $\phi_1(-\infty,\cdot,\cdot)=p^+$, and even if it means decreasing σ_0, one can assume that

$$\phi_1^\sigma>p^+-\rho\ \text{ in }(-\infty,\Sigma_0]\times\overline{\Omega},\ \text{ for all }\sigma\le\sigma_0.$$

All assumptions of Lemma 2.1 are then fulfilled, for all $\sigma\le\sigma_0$, with

$$\overline{U}(t,x,y)=U_1\left(t-\frac{\sigma}{c},x,y\right),\ \underline{U}=U_2,\ \overline{\Phi}=\phi_1^\sigma,\ \underline{\Phi}=\phi_2\text{ and }h=\Sigma_0.$$

As a consequence, $\phi_2 \le \phi_1^\sigma$ in $(-\infty, \Sigma_0] \times \overline{\Omega}$, for all $\sigma \le \sigma_0$ and, from (2.4), we finally get

$$\phi_2 \le \phi_1^\sigma \text{ in } \mathbb{R} \times \overline{\Omega} \text{ for all } \sigma \le \sigma_0.$$

Let us set

$$\sigma^* = \sup\left\{\sigma \in \mathbb{R}, \quad \phi_2 \le \phi_1^\sigma \text{ in } \mathbb{R} \times \overline{\Omega}\right\}.$$

Observe that $\sigma^* \ge \sigma_0$. Since $\phi_1(+\infty, \cdot, \cdot) = p^-$ and $\phi_2(s, x, y) > p^-(x, y)$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$, we also know that $\sigma^* < +\infty$. Moreover, $\phi_2 \le \phi_1^{\sigma^*}$ in $\mathbb{R} \times \overline{\Omega}$. Call

$$z(s, x, y) = \phi_1^{\sigma^*}(s, x, y) - \phi_2(s, x, y).$$

The function z is continuous in (s, x, y), periodic in (x, y) and nonnegative. In particular, the minimum of z over all sets of the type $[-\Sigma, \Sigma] \times \overline{\Omega}$, with $\Sigma > 0$, is reached and it is either positive or zero.

Case 1: Assume that there exists $\Sigma > 0$ such that $\min_{(s,x,y)\in[-\Sigma,\Sigma]\times\overline{\Omega}} z(s, x, y) = 0$. The function

$$v(t, x, y) := z(x \cdot e - ct, x, y)$$

is nonnegative in $\mathbb{R} \times \overline{\Omega}$ and it vanishes at a point (t^*, x^*, y^*) such that $|x^* \cdot e - ct^*| \le \Sigma$. Moreover, it satisfies the boundary condition $\nu A(x, y)\nabla v = 0$ on $\mathbb{R} \times \partial\Omega$, and the equation

$$v_t - \nabla \cdot (A\nabla v) + q \cdot \nabla v = f(x, y, U_1(t - \sigma^*/c, x, y)) - f(x, y, U_2(t, x, y))$$

in $\mathbb{R} \times \overline{\Omega}$. Since f is globally Lipschitz-continuous in $\overline{\Omega} \times \mathbb{R}$, there exists a bounded function $b(t, x, y)$ such that

$$v_t - \nabla \cdot (A\nabla v) + q \cdot \nabla v + bv = 0, \text{ for all } (t, x, y) \in \mathbb{R} \times \overline{\Omega}. \tag{2.5}$$

From the strong maximum principle and Hopf lemma, the function v is then identically 0 in $(-\infty, t^*] \times \overline{\Omega}$, and then in $\mathbb{R} \times \overline{\Omega}$ by uniqueness of the Cauchy problem associated to (2.5). We thus obtain $z \equiv 0$, that is

$$\phi_2 \equiv \phi_1^{\sigma^*} \text{ in } \mathbb{R} \times \overline{\Omega}.$$

Case 2: Assume that, for all $\Sigma > 0$, $\min_{(s,x,y)\in[-\Sigma,\Sigma]\times\overline{\Omega}} z(s, x, y) > 0$. The function z is uniformly continuous in $\mathbb{R} \times \overline{\Omega}$, thus, for all $\Sigma > 0$, there exists $\sigma_\Sigma \in (\sigma^*, \sigma^* + 1)$ such that

$$\phi_2 \le \phi_1^\sigma \text{ in } [-\Sigma, \Sigma] \times \overline{\Omega}, \quad \text{for all } \sigma \in [\sigma^*, \sigma_\Sigma]. \tag{2.6}$$

For Σ large enough, there holds

$$\phi_1^{\sigma_\Sigma} > p^+ - \rho \text{ in } (-\infty, -\Sigma] \times \overline{\Omega}.$$

Moreover, $\phi_1^{\sigma_\Sigma}(-\Sigma, x, y) \ge \phi_2(-\Sigma, x, y)$ in $\overline{\Omega}$ from (2.6). Applying Lemma 2.1 with

$$\overline{U}(t, x, y) = U_1\left(t - \frac{\sigma_\Sigma}{c}, x, y\right), \ \underline{U} = U_2, \ \overline{\Phi} = \phi_1^{\sigma_\Sigma}, \ \underline{\Phi} = \phi_2 \text{ and } h = -\Sigma,$$

we get that $\phi_2 \le \phi_1^{\sigma_\Sigma}$ in $(-\infty, -\Sigma] \times \overline{\Omega}$. Together with (2.6), since ϕ_1 is decreasing in s, it follows that

$$\exists\, \Sigma_1 > 0,\ \forall\, \Sigma \ge \Sigma_1,\ \exists\, \sigma_\Sigma > \sigma^*,\ \forall\, \sigma \in [\sigma^*, \sigma_\Sigma], \quad \phi_2 \le \phi_1^\sigma \text{ in } (-\infty, \Sigma] \times \overline{\Omega}. \tag{2.7}$$

Assume now that

$$\exists\, \varepsilon > 0,\ \exists\, \widetilde{\Sigma} > 0, \quad \phi_1^{\sigma^*} - \phi_2 \ge \varepsilon \times (\phi_2 - p^-) \text{ in } [\widetilde{\Sigma}, +\infty) \times \overline{\Omega}. \tag{2.8}$$

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $\lim_{n\to+\infty} \sigma_n = \sigma^*$. In particular, $\sigma_n > \sigma^*$ for all $n \in \mathbb{N}$. Divide (2.8) by $\phi_1^{\sigma_n} - p^-$. We get that, for all $(s, x, y) \in (-\infty, -\widetilde{\Sigma}] \times \overline{\Omega}$,

$$\frac{\phi_1^{\sigma^*}(s,x,y) - p^-(x,y)}{\phi_1^{\sigma_n}(s,x,y) - p^-(x,y)} - \frac{\phi_2(s,x,y) - p^-(x,y)}{\phi_1^{\sigma_n}(s,x,y) - p^-(x,y)} \ge \varepsilon \times \frac{\phi_2(s,x,y) - p^-(x,y)}{\phi_1^{\sigma_n}(s,x,y) - p^-(x,y)}.$$

Passing to the limit as $s \to +\infty$, it follows that $C_{[\phi_1^{\sigma^*}, \phi_1^{\sigma_n}]} - C_{[\phi_2, \phi_1^{\sigma_n}]} \ge \varepsilon \times C_{[\phi_2, \phi_1^{\sigma_n}]}$, or, equivalently,

$$\frac{1}{1+\varepsilon} \times C_{[\phi_1^{\sigma^* - \sigma_n}, \phi_1]} \ge C_{[\phi_2, \phi_1^{\sigma_n}]}.$$

But, from (2.3) applied with $\phi = \phi_1$, we know that, for n large enough, $C_{[\phi_1^{\sigma^* - \sigma_n}, \phi_1]} < 1 + \varepsilon$, whence $C_{[\phi_2, \phi_1^{\sigma_n}]} < 1$. As a consequence, there exist $n_1 \in \mathbb{N}$ and $\Sigma_2 > \widetilde{\Sigma}$ such that $\phi_2 \le \phi_1^{\sigma_{n_1}}$ in $[\Sigma_2, +\infty) \times \overline{\Omega}$, and therefore,

$$\phi_2 \le \phi_1^\sigma \text{ in } [\Sigma_2, +\infty) \times \overline{\Omega}, \quad \text{for all } \sigma \in [\sigma^*, \sigma_{n_1}]. \tag{2.9}$$

Denote $\overline{\Sigma} := \max\{\Sigma_1, \Sigma_2\}$ and $\overline{\sigma} := \min\{\sigma_{n_1}, \sigma_{\overline{\Sigma}}\}$, where $\sigma_{\overline{\Sigma}}$ is defined by (2.7). From (2.7) and (2.9), we obtain

$$\phi_2 \le \phi_1^{\overline{\sigma}} \text{ in } \mathbb{R} \times \overline{\Omega},$$

which contradicts the definition of σ^*, since $\overline{\sigma} > \sigma^*$. Therefore, the property (2.8) cannot hold.

Finally, we obtain the existence of a real number σ^* such that $\phi_1^{\sigma^*} \ge \phi_2$ and:

- either $\phi_1^{\sigma^*} \equiv \phi_2$,
- or the property (2.8) is false, thus there exists a sequence $(s_n, x_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times \overline{\Omega}$, such that $\lim_{n\to+\infty} s_n = +\infty$ and

$$0 \le \phi_1^{\sigma^*}(s_n, x_n, y_n) - \phi_2(s_n, x_n, y_n) \le \frac{\phi_2(s_n, x_n, y_n) - p^-(x_n, y_n)}{n}, \quad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

Since ϕ_1 and ϕ_2 were chosen arbitrarily, we also obtain the existence of a real number $-\sigma_*$ such that $\phi_2^{-\sigma_*} \ge \phi_1$ and:

- either $\phi_2^{-\sigma_*} \equiv \phi_1$,

- or there exists a sequence $(s'_n, x'_n, y'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$, such that $\lim_{n\to+\infty} s'_n = +\infty$ and

$$0 \le \phi_2^{-\sigma_*}(s'_n, x'_n, y'_n) - \phi_1(s'_n, x'_n, y'_n) \le \frac{\phi_1(s'_n, x'_n, y'_n) - p^-(x'_n, y'_n)}{n}, \quad \text{for all } n \in \mathbb{N}.$$

Equivalently, setting $s''_n = s'_n - \sigma_*$, we get, for all $n \in \mathbb{N}$,

$$0 \le \phi_2(s''_n, x'_n, y'_n) - \phi_1^{\sigma_*}(s''_n, x'_n, y'_n) \le \frac{\phi_1^{\sigma_*}(s''_n, x'_n, y'_n) - p^-(x'_n, y'_n)}{n}. \tag{2.11}$$

Eventually, either the property (1.8) of Theorem 2.2 holds for some $\sigma \in \mathbb{R}$, or there exist $\sigma^*,\ \sigma_* \in \mathbb{R}$ such that $\phi_1^{\sigma^*} \ge \phi_2$ and $\phi_2^{-\sigma_*} \ge \phi_1$ (that is, $\phi_2 \ge \phi_1^{\sigma_*}$) in $\mathbb{R}\times\overline{\Omega}$, and properties (2.10) and (2.11) hold true. Divide the inequalities in (2.10) and (2.11) by $\phi_1(s_n, x_n, y_n) - p^-(x_n, y_n)$ and $\phi_1(s''_n, x'_n, y'_n) - p^-(x'_n, y'_n)$ respectively, and pass to the limit as $n \to +\infty$. It follows that

$$C_{[\phi_1^{\sigma^*},\phi_1]} = C_{[\phi_2,\phi_1]} \quad \text{and} \quad C_{[\phi_2,\phi_1]} = C_{[\phi_1^{\sigma_*},\phi_1]}.$$

Thus, $C_{[\phi_1^{\sigma^*},\phi_1]} = C_{[\phi_1^{\sigma_*},\phi_1]}$. From (2.3) applied with $\phi = \phi_1$, we conclude that $\sigma^* = \sigma_* =: \sigma$. Since $\phi_1^{\sigma_*} \le \phi_2 \le \phi_1^{\sigma^*}$, we finally get that

$$\phi_1^{\sigma} \equiv \phi_2.$$

Property (1.8) has been shown. □

The assumption (2.1) in Theorem 2.2 says that, for a given speed c, any two pulsating traveling fronts have the same asymptotic behavior, up to multiplicative constants, as $s \to +\infty$, that is as they approach the unstable state p^-. This condition is essential and it is known to be fulfilled for instance in simplified situations, like in space-homogeneous settings or in straight infinite cylinders with shear flows, that is for problems (1.27) and (1.28) below. In our general periodic setting, property (2.1) is a reasonable conjecture but it has not been shown yet in general. However, in the KPP case (1.6), this property is satisfied and the proof of Theorem 1.1 follows:

Proof of Theorem 1.1. Under the KPP assumption (1.6), the hypothesis (2.1) in Theorem 2.2 is automatically fulfilled, because of formulas (1.22) and (1.23) (see Theorem 1.3 in [20]). As a consequence, (1.8) and (1.9) follow immediately. □

3 Stability of monostable fronts with speeds $c > c^*(e)$

This section is devoted to the proof of Theorem 1.3. The general strategy is based on the construction of suitable sub- and super-solutions which trap the solution u of the Cauchy problem (1.1) and which can eventually be chosen as close as we want to the front U as $t \to +\infty$. The sub- and super-solutions are close to the pulsating front U, up to some phase-shifts and exponentially small correcting terms, see Proposition 3.2 below in Subsection 3.2. Furthermore, more precise exponential estimates are established in the region where s is large, see Proposition 3.3. In Subsection 3.3, we prove a Liouville type result, that is any time-global solution which satisfies the same type of exponential estimates as in Proposition 3.3 must be a pulsating front, see Proposition 3.4. In Subsection 3.4, we complete the proof of Theorem 1.3, by arguing by contradiction and using the estimates of Subsection 3.2 and the aforementioned Liouville type result.

Due to the generality of the framework and the assumptions, the proof is rather involved and requires many technicalities. Before entering into the core of the proof, we shall first introduce in the following subsection a few notations.

3.1 Preliminary notations

We assume here that $\mu^+ > 0$ and that

$$U(t,x,y) = \phi(x\cdot e - ct, x, y)$$

is a pulsating traveling front with speed $c > c^*(e)$ satisfying (1.18).

Remember that $k(0) = \mu^- < 0$ and that $\lambda_c > 0$ is given by (1.16). By continuity of the function k, there exists then $\lambda > \lambda_c$ such that

$$-\frac{k(\lambda)}{\lambda} \;<\; c \;=\; -\frac{k(\lambda_c)}{\lambda_c} \tag{3.1}$$

and

$$k(\lambda) + \lambda c \le \mu^+. \tag{3.2}$$

Define $\omega > 0$ by

$$k(\lambda) + \lambda c = 2\,\omega. \tag{3.3}$$

Let θ be a $C^2(\overline{\Omega})$ nonpositive periodic function such that

$$\nu A\nabla\theta + \nu A e = 0 \;\text{ on } \partial\Omega. \tag{3.4}$$

For instance, up to a constant, θ can be chosen as a minimizer in H^1_{per} of the functional

$$\varphi \mapsto \int_{\Omega} \nabla\varphi A\nabla\varphi + 2\int_{\partial\Omega} (\nu A e)\,\varphi,$$

where H^1_{per} denotes the set of periodic function in Ω which are in $H^1_{loc}(\overline{\Omega})$. Let ψ^+ be given by (1.17) and $\psi = \psi_\lambda$ denote the positive principal eigenvalue of the operator L_λ, given in (1.12), such that $\|\psi\|_{L^\infty(\Omega)} = 1$. Set $m^+ = \min_{\overline{\Omega}} \psi^+ > 0$ and let $\underline{s} \in \mathbb{R}$ be such that

$$e^{-\lambda(\underline{s}-1)} \le m^+. \tag{3.5}$$

Let χ be a $C^2(\mathbb{R};[0,1])$ function such that

$$\chi'(s) \ge 0 \;\text{ for all } s\in\mathbb{R}, \quad \chi(s) = 0 \;\text{ for all } s \le \underline{s} - 1 \;\text{ and } \chi(s) = 1 \;\text{ for all } s \ge \underline{s}. \tag{3.6}$$

Let g be the function defined for all $(s,x,y) \in \mathbb{R}\times\overline{\Omega}$ by

$$g(s,x,y) = \psi(x,y)\,e^{-\lambda s}\,\chi(s+\theta(x,y)) + \psi^+(x,y)\,(1-\chi(s+\theta(x,y))).$$

Observe that g is nonnegative, bounded and periodic with respect to (x,y) in $\mathbb{R}\times\overline{\Omega}$.

Lemma 3.1 *Define*

$$\rho^+ = \min_{\overline{\Omega}} \frac{p^+ - p^-}{\psi^+} > 0. \tag{3.7}$$

There holds

$$\limsup_{\varsigma\to-\infty} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega},\ \rho\in(0,\rho^+]} \frac{\phi(s,x,y) - \rho\, g(s+\varsigma,x,y) - p^+(x,y)}{\rho\,\psi^+(x,y)} \le -1.$$

Proof. Assume the conclusion does not hold. Then, there exist $0 < \varepsilon \le 1$ and three sequences $(s_n,x_n,y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$, $(\rho'_n)_{n\in\mathbb{N}}$ in $(0,\rho^+]$ and $(\varsigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\varsigma_n = -\infty$ and

$$\frac{\phi(s_n,x_n,y_n) - \rho'_n\, g(s_n+\varsigma_n,x_n,y_n) - p^+(x_n,y_n)}{\rho'_n\,\psi^+(x_n,y_n)} \ge -1+\varepsilon$$

for all $n\in\mathbb{N}$. Up to extraction of a subsequence, either the sequence $(s_n+\varsigma_n)_{n\in\mathbb{N}}$ converges to $-\infty$ as $n\to+\infty$, or it is bounded from below. In the first case, and since $\phi\le p^+$, one has

$$\frac{-g(s_n+\varsigma_n,x_n,y_n)}{\psi^+(x_n,y_n)} \ge -1+\varepsilon.$$

The passage to the limit as $n\to+\infty$ leads to $-1\ge -1+\varepsilon$ by definition of g, which is impossible. Thus, the sequence $(s_n+\varsigma_n)_{n\in\mathbb{N}}$ is bounded from below, whence $\lim_{n\to+\infty} s_n = +\infty$. Since $g\ge 0$ and $\rho'_n\le\rho^+$, one gets that

$$\frac{\phi(s_n,x_n,y_n) - p^+(x_n,y_n)}{\psi^+(x_n,y_n)} \ge -(1-\varepsilon)\,\rho'_n \ge -(1-\varepsilon)\,\rho^+ > -\rho^+. \tag{3.8}$$

Since all functions ϕ, p^+ and ψ^+ are periodic in (x,y), one can assume that $(x_n,y_n)\to(x_\infty,y_\infty)\in\overline{\Omega}$ as $n\to+\infty$ (up to extraction of another subsequence). The limit as $n\to+\infty$ in (3.8) leads to

$$\frac{p^-(x_\infty,y_\infty) - p^+(x_\infty,y_\infty)}{\psi^+(x_\infty,y_\infty)} > -\rho^+,$$

which is ruled out by (3.7). As a consequence, Lemma 3.1 has been proved. $\square$

In the sequel, we set $s_0\le 0$ such that

$$\forall\ \rho\in(0,\rho^+],\ \forall\ (s,x,y)\in\mathbb{R}\times\overline{\Omega},\quad \frac{\phi(s,x,y) - \rho\, g(s+s_0,x,y) - p^+(x,y)}{\psi^+(x,y)} \le -\frac{\rho}{2}. \tag{3.9}$$

Set, for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$,[4]

$$\left\{\begin{array}{rcl} B(s,x,y) & = & (\zeta^-+\omega)\,\psi\, e^{-\lambda s}\,\chi(s+\theta) + (\zeta^+ + \mu^+ - \omega)\,\psi^+\,(1-\chi(s+\theta)) \\ & & +\big\{(\psi\, e^{-\lambda s} - \psi^+)\times[c + q\cdot(\nabla\theta + e) - \nabla\cdot(A\nabla\theta + Ae)] \\ & & \qquad +2\,(-\lambda\,\psi\, e^{-\lambda s}\, e - e^{-\lambda s}\,\nabla\psi + \nabla\psi^+)\,A\,(\nabla\theta+e)\big\}\,\chi'(s+\theta) \\ & & -(\psi\, e^{-\lambda s} - \psi^+)\,(\nabla\theta\, A\nabla\theta + eAe + 2\, eA\nabla\theta)\,\chi''(s+\theta) \\ C(s,x,y) & = & -\lambda\,\psi\, e^{-\lambda s}\,\chi(s+\theta) + (\psi\, e^{-\lambda s} - \psi^+)\,\chi'(s+\theta) \end{array}\right. \tag{3.10}$$

[4]In formula (3.10), when the letter e is alone, it means the direction e, while $e^{-\lambda s}$ means $\exp(-\lambda s)$.

where all functions A, q, $\zeta^\pm$, ψ, ψ^+, θ are evaluated at (x,y), and $\zeta^\pm(x,y)=\frac{\partial f}{\partial u}(x,y,p^\pm(x,y))$. Let us check that the function C is nonpositive. To see it, since $\lambda\psi\chi\ge 0$ and $\chi'\ge 0$, one only needs to check that $\psi(x,y)\,e^{-\lambda s}-\psi^+\le 0$ when $\chi'(s+\theta(x,y))>0$. If $\chi'(s+\theta(x,y))>0$, then $s+\theta(x,y)\ge \underline{s}-1$, whence $s\ge \underline{s}-1-\theta(x,y)\ge \underline{s}-1$ (θ is nonpositive) and $\psi(x,y)\,e^{-\lambda s}\le e^{-\lambda(\underline{s}-1)}\le m^+\le \psi^+(x,y)$ from (3.5). Therefore,

$$C(s,x,y)\le 0\ \text{ for all } (s,x,y)\in\mathbb{R}\times\overline{\Omega}. \tag{3.11}$$

Now, choose $\rho^->0$ such that

$$\forall\ (x,y,\rho)\in\overline{\Omega}\times[0,\rho^-],\qquad \Big|\frac{\partial f}{\partial u}(x,y,p^-(x,y)+\rho)-\zeta^-(x,y)\Big|\le\omega. \tag{3.12}$$

Remember that $\phi_s<0$ in $\mathbb{R}\times\overline{\Omega}$ and notice that, because of (1.18), (2.2) and $\lambda>\lambda_c$,

$$\sup_{(x,y)\in\overline{\Omega}}\frac{|C(s+s_0,x,y)|}{|\phi_s(s,x,y)|}\to 0\ \text{ as } s\to+\infty.$$

Owing to the definitions of the functions B and C, there exists then $s^+\ge 0$ such that

$$\forall\,(s,x,y)\in[s^+,+\infty)\times\overline{\Omega},\ \begin{cases} p^-(x,y)<\phi(s,x,y)\le p^-(x,y)+\dfrac{\rho^-}{2},\\ g(s+s_0,x,y)=\psi(x,y)\,e^{-\lambda(s+s_0)}\le\dfrac{\rho^-}{2},\\ B(s+s_0,x,y)=(\zeta^-(x,y)+\omega)\,g(s+s_0,x,y),\\ C(s+s_0,x,y)=-\lambda\,\psi(x,y)\,e^{-\lambda(s+s_0)}<0,\\ -\phi_s(s,x,y)+\rho^+\,C(s+s_0,x,y)\ge 0.\end{cases} \tag{3.13}$$

As above, one can choose $\rho_1^+\in(0,\rho^+]$ such that

$$\forall\ (x,y,\rho)\in\overline{\Omega}\times[0,\rho_1^+],\qquad \Big|\frac{\partial f}{\partial u}(x,y,p^+(x,y)-\rho\,\psi^+(x,y))-\zeta^+(x,y)\Big|\le\omega. \tag{3.14}$$

Since $\min_{\overline{\Omega}}\psi^+>0$, there exists $s^-\le 0$ such that

$$\forall\,(s,x,y)\in(-\infty,s^-]\times\overline{\Omega},\ \begin{cases} p^+(x,y)-\dfrac{\rho_1^+}{2}\,\psi^+(x,y)\le\phi(s,x,y)<p^+(x,y),\\ g(s,x,y)=\psi^+(x,y),\\ B(s,x,y)=(\zeta^+(x,y)+\mu^+-\omega)\,\psi^+(x,y),\\ C(s,x,y)=0.\end{cases} \tag{3.15}$$

Once the real numbers $s^\pm$ have been chosen, let δ be given by

$$\delta=\min_{s^-\le s\le s^+,\ (x,y)\in\overline{\Omega}}(-\phi_s(s,x,y)). \tag{3.16}$$

The real number δ is positive since the function ϕ_s is continuous, negative and periodic with respect to (x,y) in $\mathbb{R}\times\overline{\Omega}$. Define

$$\varepsilon_1=\min\left(\frac{\rho_1^+}{4},\frac{\delta}{4\,\|C\|_\infty}\right)>0 \tag{3.17}$$

and

$$\varepsilon_0=m^+\,\varepsilon_1>0, \tag{3.18}$$

where $m^+=\min_{\overline{\Omega}}\psi^+>0$.

Lastly, since the function $\frac{\partial f}{\partial u}$ is continuous in $\overline{\Omega}\times\mathbb{R}$ and periodic with respect to (x,y), the quantity

$$M=\max_{(x,y)\in\overline{\Omega},\ p^-(x,y)\le u\le p^+(x,y)}\left|\frac{\partial f}{\partial u}(x,y,u)\right| \tag{3.19}$$

is finite. Notice also that all functions g, B and C are bounded in $\mathbb{R}\times\overline{\Omega}$. Let $\underline{\sigma}$ be the nonnegative real number defined by

$$\underline{\sigma}=\max\left(\frac{M\,\|g\|_\infty+\|B\|_\infty}{\omega\,\|C\|_\infty},\frac{M\,\|g\|_\infty+\|B\|_\infty}{\omega\,\delta}\right). \tag{3.20}$$

3.2 Sub- and super-solutions

The method which is used to prove the convergence of $u(t,x,y)$ to the pulsating front $U(t,x,y)$ is first based on the construction of suitable sub- and super-solutions which converge to finite shifts of the front ϕ as $t\to+\infty$. This idea is inspired from a paper by Fife and McLeod [15] devoted to onedimensional bistable equations. The method has to be adapted here to the periodic framework and to monostable equations. Then we will prove that the shifts can be as small as we want as $x\cdot e-ct\to+\infty$. These comparisons will be used in the following subsection to prove the uniform convergence of u to the front U as $t\to+\infty$, without shift.

We assume that $\mu^+>0$ and that $U(t,x,y)=\phi(x\cdot e-ct,x,y)$ is a pulsating traveling front with speed $c>c^*(e)$ and satisfying (1.18). We use the notations of the previous section and we assume that the initial condition u_0 satisfies (1.19) and (1.20). In the sequel, for all $\kappa\in\mathbb{R}$ and $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$, we denote

$$s_\kappa(t,x)=x\cdot e-ct+\kappa-\kappa\,e^{-\omega t}.$$

Proposition 3.2 *Under all assumptions of Theorem* 1.3 *and under the above notations, there exist* $t_0>0$ *and* $\sigma_0\ge\underline{\sigma}$ *such that*

$$\begin{aligned}\max\,&[\phi(s_{\sigma_0}(t,x),x,y)-2\,\varepsilon_1\,g(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t},\ p^-(x,y)]\\ &\le u(t,x,y)\le\min\,[\phi(s_{-\sigma_0}(t,x),x,y)+g(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t},\ p^+(x,y)]\end{aligned} \tag{3.21}$$

for all $t\ge t_0$ *and* $(x,y)\in\overline{\Omega}$.

Proof. Step 1: Choice of a time $t_0 > 0$. Since u and U solve the same equation (1.1) with $p^-(x,y) \le u(t,x,y)$, $U(t,x,y) \le p^+(x,y)$ for all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$, there holds

$$|u(t,x,y) - U(t,x,y)| \le e^{Mt}\,|u_0(x,y) - U(0,x,y)| \text{ for all } (t,x,y) \in [0,+\infty) \times \overline{\Omega}, \quad (3.22)$$

where $M \in [0,+\infty)$ is defined in (3.19). In particular, it follows from (1.20) that, for each $t > 0$,

$$u(t,x,y) - p^-(x,y) = U(t,x,y) - p^-(x,y) + o(U(0,x,y) - p^-(x,y)) \text{ as } x \cdot e \to +\infty.$$

Since both U and p^- satisfy (1.1) and $U > p^-$ in $\mathbb{R} \times \overline{\Omega}$, it follows from Harnack inequality that, for each $t > 0$, there is a constant $C_t > 0$ such that

$$0 < U(0,x,y) - p^-(x,y) \le C_t\,(U(t,x,y) - p^-(x,y)) \text{ for all } (x,y) \in \overline{\Omega}.$$

As a consequence,

$$\forall\ t > 0, \quad u(t,x,y) - p^-(x,y) \sim U(t,x,y) - p^-(x,y) \text{ as } x \cdot e \to +\infty. \quad (3.23)$$

It also follows from (1.19) and (3.22) that one can choose $t_0 > 0$ small enough so that

$$\liminf_{\varsigma \to -\infty} \inf_{(x,y)\in\overline{\Omega},\ x\cdot e \le \varsigma} \frac{u(t_0,x,y) - p^+(x,y)}{\psi^+(x,y)} > -\frac{\varepsilon_0\, e^{-\omega t_0}}{m^+} = -\varepsilon_1\, e^{-\omega t_0}, \quad (3.24)$$

because of (3.18). Since $0 < 2\varepsilon_1 \le \rho_1^+/2 \le \rho^+$, it follows from (3.9) and (3.24) that

$$\begin{aligned} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} & \frac{\phi(s,x,y) - 2\,\varepsilon_1\, g(s+s_0,x,y)\, e^{-\omega t_0} - p^+(x,y)}{\psi^+(x,y)} \\ & < \liminf_{\varsigma \to -\infty} \inf_{(x,y)\in\overline{\Omega},\ x\cdot e \le \varsigma} \frac{u(t_0,x,y) - p^+(x,y)}{\psi^+(x,y)}. \end{aligned} \quad (3.25)$$

Step 2: Choice of $\sigma_0 \ge \underline{\sigma}$. We now claim that

$$\max\big[\phi(s_\sigma(t_0,x),x,y) - 2\,\varepsilon_1\, g(s_\sigma(t_0,x)+s_0,x,y)\, e^{-\omega t_0}, p^-(x,y)\big] \le u(t_0,x,y) \text{ in } \overline{\Omega} \quad (3.26)$$

for all $\sigma > 0$ large enough. Assume not. Then there exist two sequences $(x_n,y_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ and $(\sigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \sigma_n = +\infty$ and

$$\max\big[\phi(s_{\sigma_n}(t_0,x_n),x_n,y_n) - 2\,\varepsilon_1\, g(s_{\sigma_n}(t_0,x_n)+s_0,x_n,y_n)\, e^{-\omega t_0}, p^-(x_n,y_n)\big] > u(t_0,x_n,y_n)$$

for all $n \in \mathbb{N}$. Since $u \ge p^-$, one gets that

$$\phi(s_{\sigma_n}(t_0,x_n),x_n,y_n) - 2\,\varepsilon_1\, g(s_{\sigma_n}(t_0,x_n)+s_0,x_n,y_n)\, e^{-\omega t_0} > u(t_0,x_n,y_n) \quad (3.27)$$

for all $n \in \mathbb{N}$.

Up to extraction of a subsequence, two cases may occur:

either the sequence $(s_{\sigma_n}(t_0,x_n))_{n\in\mathbb{N}}$ is bounded from above, or $\lim_{n\to+\infty} s_{\sigma_n}(t_0,x_n) = +\infty$.

If it is bounded from above, then $x_n \cdot e \to -\infty$ as $n \to +\infty$. There holds

$$\frac{\phi(s_{\sigma_n}(t_0, x_n), x_n, y_n) - 2\,\varepsilon_1\, g(s_{\sigma_n}(t_0, x_n) + s_0, x_n, y_n)\, e^{-\omega t_0} - p^+(x_n, y_n)}{\psi^+(x_n, y_n)} > \frac{u(t_0, x_n, y_n) - p^+(x_n, y_n)}{\psi^+(x_n, y_n)}.$$

But the limsup of the left-hand side as $n \to +\infty$ is less than the liminf of the right-hand side, because of (3.25) and $\lim_{n\to+\infty} x_n \cdot e = -\infty$. This case is then ruled out.

Thus, $s_{\sigma_n}(t_0, x_n) \to +\infty$ as $n \to +\infty$. Since $\phi(+\infty, \cdot, \cdot) = p^-$ and $u \geq p^-$, it follows from (3.27) that

$$u(t_0, x_n, y_n) - p^-(x_n, y_n) \to 0 \ \text{ as } n \to +\infty. \tag{3.28}$$

Because of (3.24) and $0 < \varepsilon_1\, e^{-\omega t_0} < \varepsilon_1 \leq \rho^+/2 < \rho^+ = \min_{\overline{\Omega}}[(p^+ - p^-)/\psi^+]$, it follows then, as in the proof of Lemma 3.1, that the sequence $(x_n \cdot e)_{n\in\mathbb{N}}$ is bounded from below. Up to extraction of another subsequence, two subcases may occur:

either the sequence $(x_n \cdot e)_{n\in\mathbb{N}}$ is bounded, or it converges to $+\infty$ as $n \to +\infty$.

Write $x_n = x'_n + x''_n$ where $x'_n \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}$ and $(x''_n, y_n) \in C$ for all $n \in \mathbb{N}$. Up to extraction of a subsequence, one can assume that $(x''_n, y_n) \to (x_\infty, y_\infty) \in C$ as $n \to +\infty$. Set

$$u_n(t, x, y) = u_n(t, x + x'_n, y).$$

By periodicity of coefficients of (1.1), the functions u_n solve (1.1) for $t > 0$. Furthermore, $p^-(x, y) \leq u_n(t, x, y) \leq p^+(x, y)$ for all $(t, x, y) \in [0, +\infty) \times \overline{\Omega}$ and $n \in \mathbb{N}$. From standard parabolic estimates, the functions u_n converge locally uniformly in $(0, +\infty) \times \overline{\Omega}$, up to extraction of a subsequence, to a solution u_∞ of (1.1) such that

$$p^-(x, y) \leq u_\infty(t, x, y) \leq p^+(x, y) \ \text{ for all } (t, x, y) \in (0, +\infty) \times \overline{\Omega}.$$

Moreover, $u_\infty(t_0, x_\infty, y_\infty) = p^-(x_\infty, y_\infty)$ from (3.28). It follows from the strong maximum principle that $u_\infty(t, x, y) = p^-(x, y)$ for all $(t, x, y) \in (0, t_0] \times \overline{\Omega}$ (and then in $(0, +\infty) \times \overline{\Omega}$). If the sequence $(x_n \cdot e)_{n\in\mathbb{N}}$ is bounded, so is the sequence $(x'_n \cdot e)_{n\in\mathbb{N}}$, hence the function u_∞ still satisfies (3.24). This leads to a contradiction as above. Therefore, $x_n \cdot e \to +\infty$ as $n \to +\infty$. Because of (3.27), there holds

$$\frac{\phi(s_{\sigma_n}(t_0, x_n), x_n, y_n) - p^-(x_n, y_n)}{U(t_0, x_n, y_n) - p^-(x_n, y_n)} > \frac{u(t_0, x_n, y_n) - p^-(x_n, y_n)}{U(t_0, x_n, y_n) - p^-(x_n, y_n)}.$$

Because of (3.23), the right-hand side converges to 1 as $n \to +\infty$. On the other hand, the left-hand side is equal to

$$\frac{\phi(s_{\sigma_n}(t_0, x_n), x_n, y_n) - p^-(x_n, y_n)}{U(t_0, x_n, y_n) - p^-(x_n, y_n)} = \frac{\phi(x_n \cdot e - ct_0 + \sigma_n - \sigma_n e^{-\omega t_0}, x_n, y_n) - p^-(x_n, y_n)}{\phi(x_n \cdot e - ct_0, x_n, y_n) - p^-(x_n, y_n)}.$$

But property (2.2), together with $\lim_{n\to+\infty} x_n \cdot e = \lim_{n\to+\infty} \sigma_n = +\infty$, implies that the above quantity converges to 0 as $n \to +\infty$. This leads to a contradiction. Eventually, the claim (3.26) is proved.

Next, we claim that

$$u(t_0,x,y)\le\min\big[\phi(s_{-\sigma}(t_0,x),x,y)+g(s_{-\sigma}(t_0,x),x,y)\,e^{-\omega t_0},p^+(x,y)\big]\ \text{ in }\overline{\Omega}\tag{3.29}$$

for all $\sigma>0$ large enough. Assume not. Since $u\le p^+$, there exist then two sequences $(x_n,y_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ and $(\sigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\sigma_n=+\infty$ and

$$\phi(s_{-\sigma_n}(t_0,x_n),x_n,y_n)+g(s_{-\sigma_n}(t_0,x_n),x_n,y_n)\,e^{-\omega t_0}<u(t_0,x_n,y_n)$$

for all $n\in\mathbb{N}$. If $s_{-\sigma_n}(t_0,x_n)\to-\infty$ as $n\to+\infty$ up to extraction of a subsequence, then $\phi(s_{-\sigma_n}(t_0,x_n),x_n,y_n)-p^+(x_n,y_n)\to0$, while $u(t_0,x_n,y_n)\le p^+(x_n,y_n)$ and $\liminf_{n\to+\infty}g(s_{-\sigma_n}(t_0,x_n),x_n,y_n)\,e^{-\omega t_0}\ge m^+\,e^{-\omega t_0}>0$, where $m^+=\min_{\overline{\Omega}}\psi^+>0$. This gives a contradiction. Thus, the sequence $(s_{-\sigma_n}(t_0,x_n))_{n\in\mathbb{N}}$ is bounded from below, whence $x_n\cdot e\to+\infty$ as $n\to+\infty$. In particular, $u(t_0,x_n,y_n)-p^-(x_n,y_n)\to0$ as $n\to+\infty$ from (3.23). Since $\phi\ge p^-$ and $g\ge0$, one gets that $g(s_{-\sigma_n}(t_0,x_n),x_n,y_n)\to0$ as $n\to+\infty$, whence $s_{-\sigma_n}(t_0,x_n)\to+\infty$ owing to the definition of g. Moreover,

$$\frac{\phi(s_{-\sigma_n}(t_0,x_n),x_n,y_n)-p^-(x_n,y_n)}{U(t_0,x_n,y_n)-p^-(x_n,y_n)}<\frac{u(t_0,x_n,y_n)-p^-(x_n,y_n)}{U(t_0,x_n,y_n)-p^-(x_n,y_n)}\tag{3.30}$$

and the right-hand side converges to 1 as $n\to+\infty$ from (3.23). Since

$$\lim_{n\to+\infty}s_{-\sigma_n}(t_0,x_n)=\lim_{n\to+\infty}x_n\cdot e=\lim_{n\to+\infty}(x_n\cdot e-s_{-\sigma_n}(t_0,x_n))=+\infty$$

one concludes from (2.2) that the left-hand side of (3.30) converges to $+\infty$ as $n\to+\infty$. One is led to a contradiction. Hence, the claim (3.29) is proved.

In the sequel of the proof, we set a real number σ_0 large enough so that (3.26) and (3.29) are fulfilled for $\sigma=\sigma_0$, and $\sigma_0\ge\underline{\sigma}\ge0$, where $\underline{\sigma}\ge0$ has been given in (3.20).

Step 3: The lower and upper bounds in (3.21) are sub- and super-solutions of (1.1). Define

$$\mathcal{L}w=w_t-\nabla\cdot(A(x,y)\nabla w)+q(x,y)\cdot\nabla w-f(x,y,w)$$

and

$$\begin{cases}\underline{u}(t,x,y)=\phi(s_{\sigma_0}(t,x),x,y)-2\,\varepsilon_1\,g(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t},\\ \overline{u}(t,x,y)=\phi(s_{-\sigma_0}(t,x),x,y)+g(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t}\end{cases}$$

for all $(t,x,y)\in[t_0,+\infty)\times\overline{\Omega}$.

Since $\nu A\nabla U(t,\cdot,\cdot)=\nu A\nabla\psi^+=\nu A\nabla\psi-\lambda\,(\nu Ae)\,\psi=\nu A\nabla\theta+\nu Ae=0$ on $\partial\Omega$, it is immediate to see from the definitions of g and $s_{\pm\sigma_0}(t,x)$ that

$$\nu A(x,y)\nabla\underline{u}(t,x,y)=\nu A(x,y)\nabla\overline{u}(t,x,y)=0$$

for all $(t,x,y)\in[t_0,+\infty)\times\partial\Omega$.

Remember now that $p^-\le u\le p^+$ solve (1.1), and that the inequalities (3.21) are fulfilled at time t_0. In order to prove (3.21) for all $(t,x,y)\in[t_0,+\infty)\times\overline{\Omega}$, it is then enough to prove, from the maximum principle, that

$$\mathcal{L}\underline{u}\le0\ \text{ in }\Omega_-\ \text{ and }\ \mathcal{L}\overline{u}\ge0\ \text{ in }\Omega_+,$$

where

$$\begin{cases} \Omega_- = \left\{(t,x,y) \in [t_0,+\infty) \times \overline{\Omega},\ \underline{u}(t,x,y) > p^-(x,y)\right\}, \\ \Omega_+ = \left\{(t,x,y) \in [t_0,+\infty) \times \overline{\Omega},\ \overline{u}(t,x,y) < p^+(x,y)\right\}. \end{cases}$$

Let us first deal with the function $\underline{u}$. By using equations (1.1), (1.17), (3.3) and $L_\lambda \psi = k(\lambda)\psi$ in $\overline{\Omega}$, a lengthy but straightforward calculation leads to, for all $(t,x,y) \in \Omega_-$:

$$\begin{aligned} \mathcal{L}\underline{u}(t,x,y) &= f(x,y,\phi(s_{\sigma_0}(t,x),x,y)) - f(x,y,\underline{u}(t,x,y)) + \sigma_0\,\omega\,\phi_s(s_{\sigma_0}(t,x),x,y)\,e^{-\omega t} \\ &\quad -2\,\varepsilon_1\,B(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t} - 2\,\varepsilon_1\,\sigma_0\,\omega\,C(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-2\omega t}, \end{aligned}$$

where the functions B and C have been defined in (3.10).

If $(t,x,y) \in \Omega_-$ and $s_{\sigma_0}(t,x) \geq s^+$, where s^+ is given by (3.13), then

$$f(x,y,\phi(s_{\sigma_0}(t,x),x,y)) - f(x,y,\underline{u}(t,x,y)) \leq 2\,\varepsilon_1\,(\zeta^-(x,y)+\omega)\,g(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t}$$

from (3.12) and (3.13), whence

$$\begin{aligned} \mathcal{L}\underline{u}(t,x,y) &\leq 2\,\varepsilon_1\,\left[(\zeta^-(x,y)+\omega)\,g(s_{\sigma_0}(t,x)+s_0,x,y) - B(s_{\sigma_0}(t,x)+s_0,x,y)\right]\,e^{-\omega t} \\ &\quad +\sigma_0\,\omega\,\left[\phi_s(s_{\sigma_0}(t,x),x,y) - 2\,\varepsilon_1\,C(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t}\right]\,e^{-\omega t} \\ &\leq 0 \end{aligned}$$

because of (3.13) and $0 < 2\,\varepsilon_1\,e^{-\omega t} \leq \rho^+$.

If $(t,x,y) \in \Omega_-$ and $s_{\sigma_0}(t,x) \leq s^-$, where s^- is given by (3.15), then $g(s_{\sigma_0}(t,x)+s_0,x,y) = \psi^+(x,y)$ (because $s_0 \leq 0$) and

$$\begin{aligned} p^+(x,y) > \phi(s_{\sigma_0}(t,x),x,y) \geq \underline{u}(t,x,y) &\geq p^+(x,y) - \frac{\rho_1^+}{2}\,\psi^+(x,y) - 2\,\varepsilon_1\,\psi^+(x,y)\,e^{-\omega t} \\ &\geq p^+(x,y) - \rho_1^+\,\psi^+(x,y) \end{aligned}$$

because $\varepsilon_1 \leq \rho_1^+/4$ from (3.17). Thus,

$$f(x,y,\phi(s_{\sigma_0}(t,x),x,y)) - f(x,y,\underline{u}(t,x,y)) \leq 2\,\varepsilon_1\,(\zeta^+(x,y)+\omega)\,\psi^+(x,y)\,e^{-\omega t}$$

from (3.14). Since $\phi_s < 0$ and since the last two properties in (3.15) also hold with $s+s_0$ instead of s (because $s_0 \leq 0$), it follows that

$$\begin{aligned} \mathcal{L}\underline{u}(t,x,y) &\leq 2\,\varepsilon_1\,(\zeta^+(x,y)+\omega)\,\psi^+(x,y)\,e^{-\omega t} - 2\,\varepsilon_1\,(\zeta^+(x,y)+\mu^+-\omega)\,\psi^+(x,y)\,e^{-\omega t} \\ &= 2\,\varepsilon_1\,(2\,\omega-\mu^+)\,\psi^+(x,y)\,e^{-\omega t} \leq 0 \end{aligned}$$

from (3.2) and (3.3).

If $(t,x,y) \in \Omega_-$ and $s^- \leq s_{\sigma_0}(t,x) \leq s^+$, it follows from the definitions of δ, ε_1, M and $\underline{\sigma}$ in (3.16), (3.17), (3.19) and (3.20), together with the inequality $\sigma_0 \geq \underline{\sigma}$, that

$$\begin{aligned} \mathcal{L}\underline{u}(t,x,y) &\leq 2\,\varepsilon_1\,M\,\|g\|_\infty\,e^{-\omega t} + 2\,\varepsilon_1\,\|B\|_\infty\,e^{-\omega t} - \sigma_0\,\omega\,\delta\,e^{-\omega t} + 2\,\varepsilon_1\,\sigma_0\,\omega\,\|C\|_\infty\,e^{-2\omega t} \\ &\leq \frac{\delta\,(M\,\|g\|_\infty + \|B\|_\infty)\,e^{-\omega t}}{2\,\|C\|_\infty} - \frac{\sigma_0\,\omega\,\delta\,e^{-\omega t}}{2} \leq 0. \end{aligned}$$

As a conclusion, $\underline{u}$ is a sub-solution of (1.1) in Ω_-, and it is such that $\underline{u}(t_0,\cdot,\cdot) \leq u(t_0,\cdot,\cdot)$ in $\overline{\Omega}$. Thus, $\underline{u}(t,x,y) \leq u(t,x,y)$ for all $(t,x,y) \in [t_0,+\infty) \times \overline{\Omega}$ from the parabolic maximum principle.

Let us now check that $\mathcal{L}\overline{u}(t,x,y) \geq 0$ for all $(t,x,y) \in \Omega^+$. As for $\underline{u}$, it is straightforward to check that

$$\begin{aligned}
\mathcal{L}\overline{u}(t,x,y) &= f(x,y,\phi(s_{-\sigma_0}(t,x),x,y)) - f(x,y,\overline{u}(t,x,y)) - \sigma_0\,\omega\,\phi_s(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t}\\
&\quad + B(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t} - \sigma_0\,\omega\,C(s_{\sigma_0}(t,x),x,y)\,e^{-2\omega t}\\
&\geq f(x,y,\phi(s_{-\sigma_0}(t,x),x,y)) - f(x,y,\overline{u}(t,x,y)) - \sigma_0\,\omega\,\phi_s(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t}\\
&\quad + B(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t}
\end{aligned}$$

from (3.11).

If $(t,x,y) \in \Omega_+$ and $s_{-\sigma_0}(t,x) \geq s^+$, where s^+ is given by (3.13), then

$$p^-(x,y) < \phi(s_{-\sigma_0}(t,x),x,y) \leq \overline{u}(t,x,y) \leq p^-(x,y) + \rho^-$$

(notice indeed that the first four properties in (3.13) hold without s_0, since $s_0 \leq 0$). Since $\phi_s < 0$, it follows then from (3.12) and (3.13) that $\mathcal{L}\overline{u}(t,x,y) \geq 0$.

If $(t,x,y) \in \Omega_+$ and $s_{-\sigma_0}(t,x) \leq s^-$, where s^- is given by (3.15), then

$$p^+(x,y) - \rho_1^+\,\psi^+(x,y) \leq \phi(s_{-\sigma_0}(t,x),x,y) \leq \overline{u}(t,x,y) < p^+(x,y),$$

whence

$$\begin{aligned}
\mathcal{L}\overline{u}(t,x,y) \geq -(\zeta^+(x,y)+\omega)\,\psi^+(x,y)\,e^{-\omega t} + (\zeta^+(x,y)+\mu^+-\omega)\,\psi^+(x,y)\,e^{-\omega t}\\
= (\mu^+ - 2\omega)\,\psi^+(x,y)\,e^{-\omega t} \geq 0
\end{aligned}$$

from (3.2), (3.3), (3.14) and (3.15).

If $(t,x,y) \in \Omega_+$ and $s^- \leq s_{-\sigma_0}(t,x) \leq s^+$, it follows from (3.16), (3.19), (3.20) and the inequality $\sigma_0 \geq \underline{\sigma}$ that

$$\begin{aligned}
\mathcal{L}\overline{u}(t,x,y) &\geq -M\,\|g\|_\infty\,e^{-\omega t} + \sigma_0\,\omega\,\delta\,e^{-\omega t} - \|B\|_\infty\,e^{-\omega t}\\
&\geq (\underline{\sigma}\,\omega\,\delta - M\,\|g\|_\infty - \|B\|_\infty)\;e^{-\omega t} \geq 0.
\end{aligned}$$

As a conclusion, the parabolic maximum principle yields $u(t,x,y) \leq \overline{u}(t,x,y)$ for all $(t,x,y) \in [t_0,+\infty) \times \overline{\Omega}$, and the proof of Proposition 3.2 is complete. □

The following proposition states that the solution u stays close to the front ϕ when $x\cdot e - ct$ is very positive.

Proposition 3.3 *Under all assumptions of Theorem* 1.3 *and under the above notations, there exists* $\overline{\sigma} \in \mathbb{R}$ *such that, for each* $\eta > 0$*, there is* $D_\eta > 0$ *such that, for all* $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$*,*

$$\phi(x\cdot e - ct + \eta, x, y) - D_\eta\,\psi(x,y)\,e^{-\lambda(x\cdot e - ct)} \leq u(t,x,y)$$

and

$$[x\cdot e - ct \geq \overline{\sigma}] \Longrightarrow \left[u(t,x,y) \leq \phi(x\cdot e - ct - \eta, x, y) + D_\eta\,\psi(x,y)\,e^{-\lambda(x\cdot e - ct)}\right].$$

Proof. Let $t_0 > 0$ and $\sigma_0 \geq \underline{\sigma} \geq 0$ be as in Proposition 3.2. Remember that $\phi(+\infty, \cdot, \cdot) = p^-$ uniformly in $\overline{\Omega}$. It follows from (3.21) and the definition of g and χ that there exists $\sigma \in \mathbb{R}$ such that, for all $(t, x, y) \in [t_0, +\infty) \times \overline{\Omega}$ with $x \cdot e - ct \geq \sigma$, there holds

$$u(t,x,y) \leq \phi(x \cdot e - ct - \sigma_0 + \sigma_0\, e^{-\omega t}, x, y) + \psi(x,y)\, e^{-\lambda(x\cdot e - ct - \sigma_0 + \sigma_0\, e^{-\omega t})}\, e^{-\omega t} \leq p^-(x,y) + \rho^-,$$

where $\rho^- > 0$ is given in (3.12). On the other hand, for all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$,

$$u(t,x,y) \leq \phi(x \cdot e - ct, x, y) + e^{Mt}\, |u_0(x,y) - U(0,x,y)|$$

from (3.22), and $u_0(x,y) - U(0,x,y) \to 0$ uniformly as $x \cdot e \to +\infty$ from assumption (1.20). Therefore, there exists $\overline{\sigma} \geq \sigma$ such that, for all $(t,x,y) \in [0,t_0] \times \overline{\Omega}$ with $x \cdot e - ct \geq \overline{\sigma}$, there holds $u(t,x,y) \leq p^-(x,y) + \rho^-$. To sum up, for all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$, there holds

$$(x \cdot e - ct \geq \overline{\sigma}) \Longrightarrow \big(u(t,x,y) \leq p^-(x,y) + \rho^-\big)\,. \tag{3.31}$$

Let $\eta > 0$ be any positive number. We claim that

$$\phi(x \cdot e + \eta, x, y) - D\, \psi(x,y)\, e^{-\lambda\, x \cdot e} \leq u_0(x,y) \ \text{ in } \overline{\Omega} \tag{3.32}$$

for D large enough. Assume not. Then there exist two sequences $(x_n, y_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ and $(D_n)_{n\in\mathbb{N}}$ in $[0,+\infty)$ such that $\lim_{n\to+\infty} D_n = +\infty$ and

$$\phi(x_n \cdot e + \eta, x_n, y_n) - D_n\, \psi(x_n,y_n)\, e^{-\lambda\, x_n \cdot e} > u_0(x_n, y_n)$$

for all $n \in \mathbb{N}$. Since ϕ and u_0 are bounded and $\min_{\overline{\Omega}} \psi > 0$, it follows that $\lim_{n\to+\infty} x_n \cdot e = +\infty$. For all $n \in \mathbb{N}$, there holds

$$\frac{\phi(x_n \cdot e + \eta, x_n, y_n) - p^-(x_n,y_n)}{\phi(x_n \cdot e, x_n, y_n) - p^-(x_n,y_n)} > \frac{u_0(x_n,y_n) - p^-(x_n,y_n)}{\phi(x_n \cdot e, x_n, y_n) - p^-(x_n,y_n)}.$$

The right-hand side converges to 1 as $n \to +\infty$, from assumption (1.20), while the limsup of the left-hand side is not larger than $e^{-\lambda_{m,\phi}\eta} < 1$, from (2.2). One has then reached a contradiction. Hence, (3.32) holds for D large enough.

Similarly, it is easy to check that

$$u_0(x,y) \leq \phi(x \cdot e - \eta, x, y) + D\, \psi(x,y)\, e^{-\lambda\, x\cdot e} \ \text{ in } \overline{\Omega} \tag{3.33}$$

for D large enough.

In the sequel, we choose $D_\eta > 0$ such that (3.32) and (3.33) hold for $D = D_\eta$, and

$$D_\eta \geq \max\left(e^{\lambda s^+} \times \max_{\overline{\Omega}} \frac{p^+ - p^-}{\psi},\, \rho^-\, e^{\lambda\overline{\sigma}} \times \max_{\overline{\Omega}} \frac{1}{\psi}\right) \tag{3.34}$$

where s^+ and $\overline{\sigma}$ have been given in (3.13) and (3.31).

Set

$$\underline{u}_\eta(t,x,y) = \phi(x \cdot e - ct + \eta, x, y) - D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e - ct)}$$

for all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$. Notice that $\underline{u}_\eta(0,x,y) \le u(0,x,y)$ in $\overline{\Omega}$, $u \ge p^-$ and $\nu A(x,y)\nabla \underline{u}_\eta(t,x,y) = 0$ for all $(t,x,y) \in [0,+\infty) \times \partial\Omega$. In order to prove that $\underline{u}_\eta \le u$ in $[0,+\infty)\times\overline{\Omega}$, it is then sufficient to check that $\mathcal{L}\underline{u}_\eta(t,x,y) \le 0$ for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$ such that $\underline{u}_\eta(t,x,y) > p^-(x,y)$. From (1.1), (3.3) and $L_\lambda\psi = k(\lambda)\psi$ in $\overline{\Omega}$, there holds

$$\begin{array}{rcl}\mathcal{L}\underline{u}_\eta(t,x,y) & = & f(x,y,\phi(x\cdot e - ct + \eta, x, y)) - f(x,y,\underline{u}_\eta(t,x,y)) \\ & & -(2\omega + \zeta^-(x,y))\, D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e - ct)}\end{array}$$

for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$. When $\underline{u}_\eta(t,x,y) > p^-(x,y)$, then

$$D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e-ct)} < \phi(x\cdot e - ct + \eta, x, y) - p^-(x,y) < p^+(x,y) - p^-(x,y),$$

whence $D_\eta\, e^{-\lambda(x\cdot e - ct)} \le \max_{\overline{\Omega}}[(p^+ - p^-)/\psi]$. Because of (3.34), it follows that $x\cdot e - ct \ge s^+$, and then

$$\phi(x\cdot e - ct + \eta, x, y) < \phi(s^+, x, y) \le p^-(x,y) + \frac{\rho^-}{2}$$

from (3.13). In particular, when $\underline{u}_\eta(t,x,y) > p^-(x,y)$, then

$$p^-(x,y) < \underline{u}_\eta(t,x,y) < \phi(x\cdot e - ct + \eta, x, y) < p^-(x,y) + \rho^-,$$

whence

$$f(x,y,\phi(x\cdot e - ct + \eta, x, y)) - f(x,y,\underline{u}_\eta(t,x,y)) \le (\zeta^-(x,y) + \omega)\, D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e - ct)}$$

from (3.12). It follows that $\mathcal{L}\underline{u}_\eta(t,x,y) \le -\omega\, D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e-ct)} < 0$ for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$ such that $\underline{u}_\eta(t,x,y) > p^-(x,y)$. The maximum principle then yields $\underline{u}_\eta(t,x,y) \le u(t,x,y)$ for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$.

Now, set

$$\overline{u}_\eta(t,x,y) = \phi(x\cdot e - ct - \eta, x, y) + D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e - ct)}$$

for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$. Notice that $u(0,x,y) \le \overline{u}_\eta(0,x,y)$ in $\overline{\Omega}$, that $\nu A(x,y)\nabla\overline{u}_\eta(t,x,y) = 0$ for all $(t,x,y) \in [0,+\infty)\times\partial\Omega$, that

$$[x\cdot e - ct \ge \overline{\sigma}] \Longrightarrow \big[u(t,x,y) \le p^-(x,y) + \rho^-\big]$$

from (3.31), and that

$$[x\cdot e - ct = \overline{\sigma}] \Longrightarrow \big[\overline{u}_\eta(t,x,y) > p^-(x,y) + \rho^-\big]$$

from (3.34) and $\phi > p^-$. In order to prove that $u \le \overline{u}_\eta$ when $x\cdot e - ct \ge \overline{\sigma}$, it is then sufficient to check that $\mathcal{L}\overline{u}_\eta(t,x,y) \ge 0$ for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$ such that $\overline{u}_\eta(t,x,y) \le p^-(x,y) + \rho^-$. For all such (t,x,y), there holds

$$\begin{array}{rcl}\mathcal{L}\overline{u}_\eta(t,x,y) & = & f(x,y,\phi(x\cdot e - ct - \eta, x, y)) - f(x,y,\overline{u}_\eta(t,x,y)) \\ & & +(2\omega + \zeta^-(x,y))\, D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e - ct)}\end{array}$$

and $p^-(x,y) < \phi(x\cdot e - ct - \eta, x, y) < \overline{u}_\eta(t,x,y) \le p^-(x,y) + \rho^-$, whence

$$\mathcal{L}\overline{u}_\eta(t,x,y) \ge -(\zeta^-(x,y) + \omega)\, D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e-ct)} + (2\omega + \zeta^-(x,y))\, D_\eta\, \psi(x,y)\, e^{-\lambda(x\cdot e-ct)} > 0$$

from (3.12). The maximum principle yields $u(t,x,y) \le \overline{u}_\eta(t,x,y)$ for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$ such that $x\cdot e - ct \ge \overline{\sigma}$. That completes the proof of Proposition 3.3. □

3.3 A Liouville type result

The last step before the proof of Theorem 1.3 is concerned with a Liouville type result for the time-global ($t \in \mathbb{R}$) solutions of (1.1) which are trapped between two shifts of the front ϕ and which satisfy similar estimates as in Proposition 3.3, uniformly in time.

Proposition 3.4 *Under the notations of the previous subsections, let $v(t,x,y)$ be a solution of* (1.1)*, for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$, such that*

$$\forall\ (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad \phi(x \cdot e - ct + a, x, y) \le v(t,x,y) \le \phi(x \cdot e - ct + b, x, y),$$

for some $b \le 0 \le a$. Assume also that for each $\eta > 0$, there are $D_\eta > 0$ and $\sigma_\eta \in \mathbb{R}$ such that, for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$ with $x \cdot e - ct \ge \sigma_\eta$, there holds

$$\begin{aligned} &\phi(x \cdot e - ct + \eta, x, y) - D_\eta\, \psi(x,y)\, e^{-\lambda(x \cdot e - ct)} \\ &\qquad \le v(t,x,y) \le \phi(x \cdot e - ct - \eta, x, y) + D_\eta\, \psi(x,y)\, e^{-\lambda(x \cdot e - ct)}. \end{aligned} \tag{3.35}$$

Then

$$v(t,x,y) = \phi(x \cdot e - ct, x, y) = U(t,x,y) \ \ \textit{for all } (t,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Proof. Define

$$\eta^* = \min\left\{\eta \in [0,+\infty),\ v(t,x,y) \le \phi(x \cdot e - ct - \eta', x, y) \text{ in } \mathbb{R} \times \overline{\Omega} \ \text{ for all } \eta' \ge \eta\right\}.$$

The real number η^* is well-defined and it satisfies $0 \le \eta^* \le -b$, since $\phi_s < 0$ in $\mathbb{R} \times \overline{\Omega}$. Let us now prove that $\eta^* = 0$, which will imply that $u \le U$ in $\mathbb{R} \times \overline{\Omega}$. Assume that $\eta^* > 0$.

We first claim that there exists $\sigma^* \in [\sigma_{\eta^*/4}, +\infty)$ such that

$$\forall\ (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad (x \cdot e - ct \ge \sigma^*) \Longrightarrow (v(t,x,y) \le \phi(x \cdot e - ct - \eta^*/2, x, y)), \tag{3.36}$$

where the real number $\sigma_{\eta^*/4}$ is given by our assumption applied to $\eta = \eta^*/4 > 0$. If not, then there exists a sequence $(t_n, x_n, y_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \times \overline{\Omega}$ such that $s_n = x_n \cdot e - ct_n \to +\infty$ as $n \to +\infty$, and, for all $n \in \mathbb{N}$,

$$\phi(s_n - \eta^*/2, x_n, y_n) < \phi(s_n + \eta^*/4, x_n, y_n) + D_{\eta^*/4}\, \psi(x_n, y_n)\, e^{-\lambda s_n},$$

from property (3.35) applied with $\eta = \eta^*/4 > 0$. Thus,

$$1 < \frac{\phi(s_n - \eta^*/4, x_n, y_n) - p^-(x_n, y_n)}{\phi(s_n - \eta^*/2, x_n, y_n) - p^-(x_n, y_n)} + \frac{D_{\eta^*/4}\, \psi(x_n, y_n)\, e^{-\lambda s_n}}{\phi(s_n - \eta^*/2, x_n, y_n) - p^-(x_n, y_n)}$$

for all $n \in \mathbb{N}$. The limsup as $n \to +\infty$ of the first term of the right-hand side is not larger than $e^{-\lambda_{m,\phi}\eta^*/4} < 1$ from (2.2). The limit of the second term of the right-hand side is equal to 0 because of (1.18) and $\lambda > \lambda_c$. A contradiction is reached as $n \to +\infty$. Therefore, property (3.36) holds for some $\sigma^* \ge \sigma_{\eta^*/4}$.

Choose now $\sigma_* \le \sigma^*$ such that

$$\phi(s,x,y) > p^+(x,y) - \rho \ \text{ for all } (s,x,y) \in (-\infty, \sigma_*] \times \overline{\Omega}, \tag{3.37}$$

where $\rho > 0$ is given in (1.5).

We then claim that

$$\inf_{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ \sigma_*\le x\cdot e-ct\le\sigma^*} \phi(x\cdot e-ct-\eta^*,x,y)-v(t,x,y)>0. \tag{3.38}$$

Notice first that $v(t,x,y)\le\phi(x\cdot e-ct-\eta^*,x,y)$ in $\mathbb{R}\times\overline{\Omega}$ by definition of η^*. Assume that the claim (3.38) is not true. Then there exists a sequence $(t_n,x_n,y_n)_{n\in\mathbb{N}}$ such that $s_n=x_n\cdot e-ct_n\in[\sigma_*,\sigma^*]$ for all $n\in\mathbb{N}$, and

$$\phi(x_n\cdot e-ct_n-\eta^*,x_n,y_n)-v(t_n,x_n,y_n)\to 0\ \text{ as } n\to+\infty. \tag{3.39}$$

For each $n\in\mathbb{N}$, write $x_n=x_n'+x_n''$, where $x_n'\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}$ and $(x_n'',y_n)\in C$, and

$$v_n(t,x,y)=v(t+t_n,x+x_n',y).$$

Up to extraction of a subsequence, one can assume that, as $n\to+\infty$, $s_n\to s_\infty\in[\sigma_*,\sigma^*]$, $(x_n'',y_n)\to(x_\infty,y_\infty)\in C$ and $v_n(t,x,y)\to v_\infty(t,x,y)$ locally uniformly in (t,x,y), where v_∞ solves (1.1) in $\mathbb{R}\times\overline{\Omega}$. There holds $v_n(t,x,y)\le\phi(x\cdot e-ct+x_n'\cdot e-ct_n-\eta^*,x,y)$ since ϕ is periodic in (x,y), whence

$$v_\infty(t,x,y)\le\phi(x\cdot e-ct+s_\infty-x_\infty\cdot e-\eta^*,x,y)$$

for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$. Furthermore, $v_\infty(0,x_\infty,y_\infty)=\phi(s_\infty-\eta^*,x_\infty,y_\infty)$ from (3.39). Hence,

$$v_\infty(t,x,y)=\phi(x\cdot e-ct+s_\infty-x_\infty\cdot e-\eta^*,x,y)\ \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega}, \tag{3.40}$$

from the strong maximum principle and periodicity of ϕ in the variables (x,y). On the other hand, if $x\cdot e-ct\ge\sigma^*+ct_n-x_n'\cdot e$, then

$$v_n(t,x,y)\le\phi((x+x_n')\cdot e-c(t+t_n)-\eta^*/2,x,y)$$

from (3.36), whence

$$[x\cdot e-ct\ge\sigma^*-s_\infty+x_\infty\cdot e]\Longrightarrow[v_\infty(t,x,y)\le\phi(x\cdot e-ct+s_\infty-x_\infty\cdot e-\eta^*/2,x,y)].$$

This contradicts (3.40), since $\phi_s<0$ and $\eta^*>0$.

Therefore, property (3.38) holds. By continuity and (x,y)-periodicity of ϕ, there exists then η_* such that $\eta^*/2\le\eta_*<\eta^*$ and, for all $\eta\in[\eta_*,\eta^*]$,

$$(\sigma_*\le x\cdot e-ct\le\sigma^*)\Longrightarrow(v(t,x,y)\le\phi(x\cdot e-ct-\eta,x,y)).$$

Actually, the previous inequality also holds when $x\cdot e-ct\ge\sigma^*$, because of (3.36) and $\phi_s<0$. Pick any η in $[\eta_*,\eta^*]$ $(\subset[0,\eta^*])$. In the region where $x\cdot e-ct\le\sigma_*$, then $\phi(x\cdot e-ct-\eta,x,y)>p^+(x,y)-\rho$, from (3.37) and $\phi_s<0$. All assumptions of Lemma 2.1 are satisfied with $h=\sigma_*$, $\overline{U}(t,x,y)=\phi(x\cdot e-ct-\eta,x,y)$, $\overline{\Phi}=\phi(\cdot--\eta,\cdot,\cdot)$, $\underline{U}=v$ and $\underline{\Phi}(s,x,y)=v((x\cdot e-s)/c,x,y)$, apart from the fact that $\underline{\Phi}$ may not be periodic in (x,y).

However, since $\underline{\Phi} \le \phi(\cdot + b, \cdot, \cdot) < p^+$, the arguments used in the proof of Lemma 2.1 (that is Lemma 2.3 of [20]) can be immediately extended to the present case. They yield the inequality

$$v(t,x,y) \le \phi(x\cdot e - ct - \eta, x, y) \text{ for all } (t,x,y) \text{ such that } x\cdot e - ct \le \sigma_*.$$

Eventually, $v(t,x,y) \le \phi(x\cdot e - ct - \eta, x, y)$ in $\mathbb{R}\times\overline{\Omega}$ for all $\eta \in [\eta_*, \eta^*]$. Since $\eta_* < \eta^*$, that contradicts the minimality of η^*. As a conclusion η^* cannot be positive, which proves that $v(t,x,y) \le \phi(x\cdot e - ct, x, y)$ in $\mathbb{R}\times\overline{\Omega}$.

The proof of the opposite inequality is exactly similar. Finally, $v(t,x,y) = \phi(x\cdot e - ct, x, y)$ in $\mathbb{R}\times\overline{\Omega}$, which is the desired result. □

Remark 3.5 Notice that the two key tools in the proof of Proposition 3.4 are first the property (2.2), which holds for any pulsating front in the sense of (1.7), and second the fact that $e^{-\lambda s} = o(\phi(s,x,y) - p^-(x,y))$ as $s \to +\infty$, uniformly in $(x,y) \in \overline{\Omega}$.

3.4 Proof of Theorem 1.3

With the results of the previous subsections, we are now able to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Assume that the limit (1.21) does not hold. Then there exist $\varepsilon > 0$ and a sequence $(t_n, x_n, y_n)_{n\in\mathbb{N}}$ in $[0,+\infty)\times\overline{\Omega}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $|u(t_n,x_n,y_n) - U(t_n,x_n,y_n)| \ge \varepsilon$ for all $n \in \mathbb{N}$, that is

$$|u(t_n,x_n,y_n) - \phi(s_n,x_n,y_n)| \ge \varepsilon, \tag{3.41}$$

where $s_n = x_n\cdot e - ct_n$. Under the notations of Proposition 3.2, and using the monotonicity of ϕ in s, there holds

$$\phi(s_n + \sigma_0, x_n, y_n) - 2\,\varepsilon_1\,\|g\|_\infty\, e^{-\omega t_n} \le u(t_n,x_n,y_n) \le \phi(s_n - \sigma_0, x_n, y_n) + \|g\|_\infty\, e^{-\omega t_n}.$$

If $s_n \to -\infty$, up to extraction of a subsequence, then

$$\phi(s_n + \sigma_0, x_n, y_n) - p^+(x_n,y_n) - 2\,\varepsilon_1\,\|g\|_\infty\, e^{-\omega t_n} \le u(t_n,x_n,y_n) - p^+(x_n,y_n) \le 0,$$

whence $\lim_{n\to+\infty} u(t_n,x_n,y_n) - p^+(x_n,y_n) = 0 = \lim_{n\to+\infty} \phi(s_n,x_n,y_n) - p^+(x_n,y_n)$. This contradicts (3.41). Therefore, the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded from below. Similarly, one can prove that it is bounded from above.

For each $n \in \mathbb{N}$, write $x_n = x_n' + x_n''$, where $x_n' \in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}$ and $(x_n'', y_n) \in C$. Up to extraction of a subsequence, one can assume that $s_n \to s_\infty \in \mathbb{R}$, $(x_n'', y_n) \to (x_\infty, y_\infty) \in C$ as $n \to +\infty$. Set $t_n' = t_n + (s_\infty - x_\infty\cdot e)/c$ and observe that $x_n'\cdot e - ct_n' \to 0$ as $n \to +\infty$. Denote

$$u_n(t,x,y) = u(t + t_n', x + x_n', y).$$

Up to extraction of another subsequence, the functions u_n converge locally uniformly in $\mathbb{R}\times\overline{\Omega}$ to a time-global solution u_∞ of (1.1) in $\mathbb{R}\times\overline{\Omega}$. Furthermore, Proposition 3.2 implies that, for each $n\in\mathbb{N}$ and $(t,x,y)\in[-t'_n,+\infty)\times\overline{\Omega}$,

$$\begin{array}{l}\phi(x\cdot e-ct+x'_n\cdot e-ct'_n+\sigma_0-\sigma_0\,e^{-\omega(t+t'_n)},x,y)-2\,\varepsilon_1\,\|g\|_\infty\,e^{-\omega(t+t'_n)}\\ \qquad\le u_n(t,x,y)\le\phi(x\cdot e-ct+x'_n\cdot e-ct'_n-\sigma_0+\sigma_0\,e^{-\omega(t+t'_n)},x,y)+\|g\|_\infty\,e^{-\omega(t+t'_n)},\end{array}$$

whence

$$\phi(x\cdot e-ct+\sigma_0,x,y)\le u_\infty(t,x,y)\le\phi(x\cdot e-ct-\sigma_0,x,y)\tag{3.42}$$

for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$.

Let $\overline{\sigma}\in\mathbb{R}$ be as in Proposition 3.3. It follows that for each $\eta>0$, there is $D_\eta>0$ such that, for each $n\in\mathbb{N}$ and $(t,x,y)\in[-t'_n,+\infty)\times\overline{\Omega}$, there holds

$$\phi(x\cdot e-ct+x'_n\cdot e-ct'_n+\eta,x,y)-D_\eta\,\psi(x,y)\,e^{-\lambda(x\cdot e-ct+x'_n\cdot e-ct'_n)}\le u_n(t,x,y)$$

and

$$\begin{array}{l}[(x+x'_n)\cdot e-c(t+t'_n)\ge\overline{\sigma}]\Longrightarrow\\ \qquad\big[u_n(t,x,y)\le\phi(x\cdot e-ct+x'_n\cdot e-ct'_n-\eta,x,y)+D_\eta\,\psi(x,y)\,e^{-\lambda(x\cdot e-ct+x'_n\cdot e-ct'_n)}\big].\end{array}$$

The passage to the limit as $n\to+\infty$ yields, for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$ and $\eta>0$,

$$\left\{\begin{array}{l}\phi(x\cdot e-ct+\eta,x,y)-D_\eta\,\psi(x,y)\,e^{-\lambda(x\cdot e-ct)}\le u_\infty(t,x,y),\\ {[x\cdot e-ct\ge\overline{\sigma}]}\Longrightarrow\big[u_\infty(t,x,y)\le\phi(x\cdot e-ct-\eta,x,y)+D_\eta\,\psi(x,y)\,e^{-\lambda(x\cdot e-ct)}\big].\end{array}\right.\tag{3.43}$$

It finally follows from (3.42) and (3.43) and Proposition 3.4 that $u_\infty(t,x,y)=\phi(x\cdot e-ct,x,y)$ in $\mathbb{R}\times\overline{\Omega}$ (we here apply a particular case of Proposition 3.4, that is when the real numbers σ_η can all be set to $\overline{\sigma}$, independently of $\eta>0$). But assumption (3.41) implies that $|u_n(t_n-t'_n,x''_n,y_n)-\phi(s_n,x''_n,y_n)|\ge\varepsilon$, whence

$$|u_\infty((-s_\infty+x_\infty\cdot e)/c,x_\infty,y_\infty)-\phi(s_\infty,x_\infty,y_\infty)|\ge\varepsilon.$$

One has reached a contradiction. Hence, formula (1.21) is proved and the proof of Theorem 1.3 is complete. □

4 Stability of KPP fronts with speeds $c^*(e)$

This section is devoted to the proof of Theorem 1.5, under the KPP condition (1.6). Actually, because of (1.22) when $c>c^*(e)$, part 1) is an immediate consequence of Theorem 1.3. Only part 2) on the stability of KPP fronts with minimal speeds $c^*(e)$ remains to be proved. The proof follows the main scheme as that of Theorem 1.3. However, the ideas and the stability result are new even in the special cases which were previously considered in the literature. Two additional serious difficulties arise: firstly the sub- and super-solutions must take into account the fact that the behavior of the KPP fronts with minimal speeds $c^*(e)$ as they approach p^- is not purely exponential $e^{-\lambda^* s}$, secondly, because of the criticality of λ^*, some of the ideas used in Section 3 cannot just be adapted (for instance, there is no λ satisfying (3.1) with $\lambda_c = \lambda^*$). The sub- and super-solutions involve products of exponentially decaying functions and suitable polynomial factors which are given in terms of some derivatives of the principal eigenfunctions ψ_λ with respect to λ at $\lambda = \lambda^*$.

Proof of part 2) of Theorem 1.5. Up to a shift in time, we can then assume that $B = B_\phi$ in assumption (1.26), that is $u_0(x,y) - p^-(x,y) \sim U(0,x,y) - p^-(x,y)$ as $x\cdot e \to +\infty$, where $B_\phi > 0$ is given in formula (1.23).

Step 1: Choice of parameters. Since U is a pulsating front in the sense of (1.7) with speed $c^*(e)$, it follows from [20], as already underlined, that there exists a unique $\lambda^* > 0$ such that $k(\lambda^*)+c^*(e)\lambda^* = 0$ and λ^* is a root of $k(\lambda)+\lambda c = 0$ with multiplicity $2m+2$. Furthermore, the function $\lambda \mapsto k(\lambda)$ is analytic (see [12, 28]) and, because of the normalization condition (1.14) and standard elliptic estimates, the principal eigenfunctions ψ_λ of the operators L_λ given in (1.12) are also analytic with respect to λ in the spaces $C^{2,\alpha}(\overline{\Omega})$. For each $j \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, call $\psi_\lambda^{(j)}$ the j-th order derivative of ψ_λ with respect to λ, under the convention that $\psi_\lambda^{(0)} = \psi_\lambda$. All these functions are periodic and of class C^2 in $\overline{\Omega}$. Denote $L_\lambda^{(j)}$ the operator whose coefficients are the j-th order derivatives with respect to λ of the coefficients of L_λ. In other words,

$$L_\lambda^{(0)}\psi = L_\lambda\psi,\ \ L_\lambda^{(1)}\psi = 2\,eA\nabla\psi + [\nabla\cdot(Ae) - q\cdot e - 2\,\lambda\,eAe]\psi,\ \ L_\lambda^{(2)}\psi = -2\,eAe\,\psi$$

and $L_\lambda^{(j)}\psi = 0$ for all $j \geq 3$ and for all $\psi \in C^2(\overline{\Omega})$ and $\lambda \in \mathbb{R}$. Differentiating the relation $L_\lambda\psi_\lambda - k(\lambda)\psi_\lambda = 0$ with respect to λ yields

$$\begin{cases} L_\lambda\psi_\lambda^{(1)} - k(\lambda)\psi_\lambda^{(1)} + 2\,eA\nabla\psi_\lambda + [\nabla\cdot(Ae) - q\cdot e - 2\,\lambda\,eAe]\psi_\lambda - k'(\lambda)\psi_\lambda \\ = (L_\lambda - k(\lambda))\psi_\lambda^{(1)} + (L_\lambda^{(1)} - k'(\lambda))\psi_\lambda = 0, \\ L_\lambda\psi_\lambda^{(j)} - k(\lambda)\psi_\lambda^{(j)} + j\left(2\,eA\nabla\psi_\lambda^{(j-1)} + [\nabla\cdot(Ae) - q\cdot e - 2\,\lambda\,eAe]\,\psi_\lambda^{(j-1)}\right) \\ -j\ k'(\lambda)\psi_\lambda^{(j-1)} - 2\,\mathrm{C}_j^2\,eAe\,\psi_\lambda^{(j-2)} - \displaystyle\sum_{2\leq l\leq j}\mathrm{C}_j^l\,k^{(l)}(\lambda)\,\psi_\lambda^{(j-l)} \\ = (L_\lambda - k(\lambda))\psi_\lambda^{(j)} + j(L_\lambda^{(1)} - k'(\lambda))\psi_\lambda^{(j-1)} + \mathrm{C}_j^2\,L_\lambda^{(2)}\psi_\lambda^{(j-2)} - \displaystyle\sum_{2\leq l\leq j}\mathrm{C}_j^l\,k^{(l)}(\lambda)\,\psi_\lambda^{(j-l)} \\ = 0 \quad \text{for all } j \geq 2, \end{cases} \tag{4.1}$$

where $C_n^m = n!/(m!(n-m)!)$ for all integers m, n such that $m \leq n$. Similarly, since $\nu A\nabla\psi_\lambda = \lambda\,(\nu Ae)\,\psi_\lambda$ on $\partial\Omega$ for all $\lambda \in \mathbb{R}$, one gets that, for all $\lambda \in \mathbb{R}$,

$$\nu A\nabla\psi_\lambda^{(j)} - \lambda\,(\nu Ae)\,\psi_\lambda^{(j)} - j\,(\nu Ae)\psi_\lambda^{(j-1)} = 0 \text{ on } \partial\Omega,\ \text{ for all } j \geq 1. \tag{4.2}$$

Let i and I be the functions defined by

$$\begin{cases} i(s,x,y) &=& B_\phi\,e^{-\lambda^* s} \times \left[\displaystyle\sum_{j=0}^{2m+1}(-1)^j\,\mathrm{C}_{2m+1}^j\,s^{2m+1-j}\,\psi_{\lambda^*}^{(j)}(x,y)\right], \\ I(t,x,y) &=& i(x\cdot e - c^*(e)t, x, y). \end{cases}$$

Notice that

$$i(s,x,y) \sim B_\phi\, e^{-\lambda^* s}\, s^{2m+1}\, \psi_{\lambda^*}(x,y) \sim \phi(s,x,y) - p^-(x,y) \ \text{ as } s \to +\infty, \tag{4.3}$$

uniformly in $(x,y) \in \overline{\Omega}$, from (1.23) and the fact that $\min_{\overline{\Omega}} \psi_{\lambda^*} > 0$. It also follows from (4.1) and (4.2) applied to $\lambda = \lambda^*$ that $\nu A(x,y)\nabla I(t,x,y) = 0$ for all $(t,x,y) \in \mathbb{R} \times \partial\Omega$, and

$$I_t - \nabla \cdot (A(x,y)\nabla I) + q(x,y) \cdot \nabla I - \zeta^-(x,y)\, I = 0 \ \text{ in } \mathbb{R} \times \overline{\Omega}. \tag{4.4}$$

Now, for $\mu \in \mathbb{R}$ and $a > 0$, denote

$$\left\{\begin{array}{rcl} h(s,x,y) & = & e^{-\mu s} \times \left[\displaystyle\sum_{j=0}^{2m+2} (-1)^j\, \mathrm{C}_{2m+2}^j\, (s+a)^{2m+2-j}\, \psi_\mu^{(j)}(x,y) \right], \\ H(t,x,y) & = & h(x \cdot e - c^*(e)t, x, y). \end{array}\right.$$

Because of (4.2), there holds $\nu A(x,y)\nabla H(t,x,y) = 0$ for $(t,x,y) \in \mathbb{R} \times \partial\Omega$. Notice that

$$h(s,x,y) \sim e^{-\mu s}\, s^{2m+2}\, \psi_\mu(x,y) \ \text{ as } s \to +\infty, \ \text{ uniformly in } (x,y) \in \overline{\Omega}. \tag{4.5}$$

It also follows from the definition of m and from the proof of Proposition 4.5 of [20] that one can choose $\mu - \lambda^* > 0$ small enough and $a > 0$ large enough so that

$$\left\{\begin{array}{l} h_s(s,x,y) \le 0 \ \text{ for all } (s,x,y) \in [0,+\infty) \times \overline{\Omega}, \\ H_t - \nabla \cdot (A(x,y)\nabla H) + q(x,y) \cdot \nabla H - \zeta^-(x,y)\, H \le -\upsilon\, e^{-\mu s}\, (s+a)^{2m+2} < 0 \end{array}\right. \tag{4.6}$$

for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$ such that $s = x \cdot e - c^*(e)t \ge 0$, where

$$\upsilon = \frac{|k^{(2m+2)}(\lambda^*)|\, \kappa^*\, (\mu - \lambda^*)^{2m+2}}{4\,(2m+2)!} > 0 \ \text{ and } \kappa^* = \min_{\overline{\Omega}}\ \psi_{\lambda^*} > 0. \tag{4.7}$$

Even if it means decreasing $\mu - \lambda^*$, one can also assume without loss of generality that

$$\lambda^* < \mu < \lambda^*(1+\beta), \tag{4.8}$$

where one recalls that $\beta > 0$ is such that the function $(x,y,u) \mapsto \frac{\partial f}{\partial u}(x,y,p^-(x,y)+u)$ is of class $C^{0,\beta}(\overline{\Omega} \times [0,\gamma])$, for some $\gamma > 0$.

Let θ be a $C^2(\overline{\Omega})$ nonpositive periodic function satisfying (3.4). Let ψ^+ be given by (1.17), and denote $m^+ = \min_{\overline{\Omega}} \psi^+ > 0$. Because of (1.23) and $\mu > \lambda^*$, one can choose a real number $\underline{s}$ such that $\underline{s} \ge 1$ and

$$0 \le h(s,x,y) \le \frac{\phi(s,x,y) - p^-(x,y)}{2} \le m^+ \ \text{ for all } (s,x,y) \in [\underline{s} - 1, +\infty) \times \overline{\Omega}.$$

Let $\chi \in C^2(\mathbb{R};[0,1])$ be as in (3.6) and let g be the function defined in $\mathbb{R} \times \overline{\Omega}$ by

$$g(s,x,y) = -h(s,x,y)\, \chi(s + \theta(x,y)) + \psi^+(x,y)\, (1 - \chi(s + \theta(x,y))).$$

Notice that $\chi(s+\theta(x,y))=0$ for all $(s,x,y)\in(-\infty,\underline{s}-1]\times\overline{\Omega}$, and that g is bounded, C^2 in $\mathbb{R}\times\overline{\Omega}$ and periodic with respect to the variables (x,y). Furthermore, $g\ge -m^+$ in $\mathbb{R}\times\overline{\Omega}$, and, for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$,

$$\left\{\begin{array}{lll} s\le \underline{s}-1 & \Longrightarrow & g(s,x,y)=\psi^+(x,y)\ge 0,\\ s\ge \underline{s}-1 & \Longrightarrow & g(s,x,y)\ge -h(s,x,y)\ge -\dfrac{\phi(s,x,y)-p^-(x,y)}{2}.\end{array}\right. \tag{4.9}$$

We then claim that

$$\limsup_{\varsigma\to-\infty}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega},\ \rho\in(0,\rho^+/2]}\ \frac{\phi(s,x,y)-\rho\,g(s+\varsigma,x,y)-p^+(x,y)}{\rho\,\psi^+(x,y)}\le -1,$$

where $\rho^+=\min_{\overline{\Omega}}[(p^+-p^-)/\psi^+]>0$. Assume not. There exist then $0<\varepsilon\le 1$ and three sequences $(s_n,x_n,y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$, $(\rho_n')_{n\in\mathbb{N}}$ in $(0,\rho^+/2]$ and $(\varsigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\varsigma_n=-\infty$ and

$$\frac{\phi(s_n,x_n,y_n)-\rho_n'\,g(s_n+\varsigma_n,x_n,y_n)-p^+(x_n,y_n)}{\rho_n'\,\psi^+(x_n,y_n)}\ge -1+\varepsilon$$

for all $n\in\mathbb{N}$. As in the proof of Lemma 3.1, it follows that the sequence $(s_n+\varsigma_n)_{n\in\mathbb{N}}$ is bounded from below, whence $\lim_{n\to+\infty}s_n=+\infty$. The last part of the argument is different from that of Lemma 3.1, since g is not nonnegative anymore. For each $n\in\mathbb{N}$, there holds

$$\begin{array}{rcl}\dfrac{\rho^+}{2}+\dfrac{\phi(s_n,x_n,y_n)-p^+(x_n,y_n)}{\psi^+(x_n,y_n)} & \ge & \dfrac{\rho_n'm^+}{\psi^+(x_n,y_n)}+\dfrac{\phi(s_n,x_n,y_n)-p^+(x_n,y_n)}{\psi^+(x_n,y_n)}\\ & \ge & -(1-\varepsilon)\rho_n'\ge -(1-\varepsilon)\,\dfrac{\rho^+}{2}>-\dfrac{\rho^+}{2}\end{array}$$

since $0<\rho_n'\le\rho^+/2$, and $g\ge -m^+$ in $\mathbb{R}\times\overline{\Omega}$. This leads to a contradiction as in Lemma 3.1. Therefore, one can choose $s_0\le 0$ such that

$$\frac{\phi(s,x,y)-\rho\,g(s+s_0,x,y)-p^+(x,y)}{\psi^+(x,y)}\le -\frac{\rho}{2}$$

for all $\rho\in(0,\rho^+/2]$ and $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$.

Set

$$G(t,x,y)=g(x\cdot e-c^*(e)t,x,y)\,e^{-\omega t}$$

for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$, where

$$\omega=\frac{\mu^+}{2}>0$$

and $\mu^+>0$ is given in (1.17). The function G is of class $C^2(\mathbb{R}\times\overline{\Omega})$ and, because of (4.6), there exists a continuous, bounded and (x,y)-periodic function B in $\mathbb{R}\times\overline{\Omega}$ such that

$$G_t-\nabla\cdot(A(x,y)\nabla G)+q(x,y)\cdot\nabla G=B(x\cdot e-c^*(e)t,x,y)\,e^{-\omega t}, \tag{4.10}$$

where

$$\left\{\begin{array}{ll} B(s,x,y)\ge(-\zeta^-(x,y)+\omega)\,h(s,x,y) & \text{if } s\ge\underline{s}+\|\theta\|_\infty\ (\ge 0),\\ B(s,x,y)=(\zeta^+(x,y)+\mu^+-\omega)\,\psi^+(x,y) & \text{if } s\le\underline{s}-1.\end{array}\right.$$

For all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$, define

$$C(s,x,y)=h_s(s,x,y)\,\chi(s+\theta(x,y))-(h(s,x,y)+\psi^+(x,y))\,\chi'(s+\theta(x,y)).$$

The function C is continuous and bounded in $\mathbb{R}\times\overline{\Omega}$, and periodic in the variables (x,y). Observe that $C(s,x,y)\sim-\mu e^{-\mu s}s^{2m+2}\psi_\mu(x,y)$ as $s\to+\infty$, uniformly in $(x,y)\in\overline{\Omega}$, whence $C(s,x,y)=o(-\phi_s(s,x,y))$ as $s\to+\infty$ from (1.23), (2.2) and $\mu>\lambda^*$. Choose $\rho^->0$ as in (3.12) and $s^+\geq 0$ such that

$$\forall\ (s,x,y)\in[s^+,+\infty)\times\overline{\Omega},\qquad \begin{cases} p^-(x,y)<\phi(s,x,y)\leq p^-(x,y)+\dfrac{\rho^-}{2},\\ -\dfrac{\rho^-}{2}\leq g(s+s_0,x,y)=-h(s+s_0,x,y)\leq 0,\\ B(s+s_0,x,y)\geq(-\zeta^-(x,y)+\omega)\,h(s+s_0,x,y),\\ C(s+s_0,x,y)=h_s(s+s_0,x,y)\leq 0,\\ |C(s,x,y)|\leq-\phi_s(s,x,y).\end{cases}\tag{4.11}$$

Lastly, define

$$M=\max_{(x,y)\in\overline{\Omega},\ p^-(x,y)-\|g\|_\infty\leq u\leq p^+(x,y)+\|g\|_\infty}\left|\frac{\partial f}{\partial u}(x,y,u)\right|\geq 0,\tag{4.12}$$

let $\rho_1^+>0$, $s^-\leq 0$ and $\delta>0$ be given as in (3.14), (3.15) and (3.16), and set

$$\varepsilon_1=\min\left(\frac{\rho_1^+}{4},\frac{\delta}{4\,\|C\|_\infty},\frac{1}{2}\right)>0,\quad \varepsilon_0=m^+\varepsilon_1>0\ \text{ and }\ \underline{\sigma}=\frac{M\,\|g\|_\infty+\|B\|_\infty}{\omega\,\|C\|_\infty}\geq 0.\tag{4.13}$$

Step 2: Comparison with sub- and super-solutions. Assume now that the initial condition u_0 satisfies (1.19) and (1.20). For all $(t,x,y)\in[0,+\infty)\times\overline{\Omega}$, define

$$\begin{cases}\underline{u}(t,x,y)&=&\phi(s_{\sigma_0}(t,x),x,y)-2\,\varepsilon_1\,g(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t},\\ \overline{u}(t,x,y)&=&\phi(s_{-\sigma_0}(t,x),x,y)+2\,\varepsilon_1\,g(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t},\end{cases}$$

where $s_\kappa(t,x)=x\cdot e-c^*(e)t+\kappa-\kappa e^{-\omega t}$ and σ_0 shall be chosen below.

As in Step 1 of the proof of Proposition 3.2, one can choose $t_0>0$ such that (3.25) holds. Then we claim that

$$\max\left[\underline{u}(t_0,x,y),p^-(x,y)\right]\leq u(t_0,x,y)\ \text{ in }\overline{\Omega},\tag{4.14}$$

for σ_0 large enough. If not, there exist two sequences $(x_n,y_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ and $(\sigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\sigma_n=+\infty$ and

$$\phi(s_{\sigma_n}(t_0,x_n),x_n,y_n)-2\,\varepsilon_1\,g(s_{\sigma_n}(t_0,x_n)+s_0,x_n,y_n)\,e^{-\omega t_0}>u(t_0,x_n,y_n)$$

for all $n\in\mathbb{N}$. As in Step 2 of the proof of Proposition 3.2, it follows that $\lim_{n\to+\infty}s_{\sigma_n}(t_0,x_n)=+\infty$ and $\lim_{n\to+\infty}x_n\cdot e=+\infty$. There holds

$$\frac{\phi(s_{\sigma_n}(t_0,x_n),x_n,y_n)-p^-(x_n,y_n)}{U(t_0,x_n,y_n)-p^-(x_n,y_n)}-\frac{2\,\varepsilon_1\,g(s_{\sigma_n}(t_0,x_n)+s_0,x_n,y_n)\,e^{-\omega t_0}}{U(t_0,x_n,y_n)-p^-(x_n,y_n)}$$
$$>\frac{u(t_0,x_n,y_n)-p^-(x_n,y_n)}{U(t_0,x_n,y_n)-p^-(x_n,y_n)}.$$

The right-hand side converges to 1 as $n \to +\infty$, from (3.23), while the first term of the left-hand side converges to 0, from (2.2). Lastly, the second term in the left-hand converges to 0 too, from (1.23) and $\mu > \lambda^*$. This leads to a contradiction, which yields (4.14) for σ_0 large enough.

Similarly, there holds

$$u(t_0,x,y) \le \min\big[\overline{u}(t_0,x,y), p^+(x,y)\big] \quad \text{in } \overline{\Omega}, \tag{4.15}$$

for σ_0 large enough. Assume not. Then there exist two sequences $(x_n,y_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ and $(\sigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\sigma_n = +\infty$ and

$$\phi(s_{-\sigma_n}(t_0,x_n),x_n,y_n) + 2\,\varepsilon_1\, g(s_{-\sigma_n}(t_0,x_n),x_n,y_n)\, e^{-\omega t_0} < u(t_0,x_n,y_n)$$

for all $n \in \mathbb{N}$. As in Step 2 of the proof of Proposition 3.2, it follows that the sequence $(s_{-\sigma_n}(t_0,x_n))_{n\in\mathbb{N}}$ is bounded from below, whence $\lim_{n\to+\infty} x_n\cdot e = +\infty$ and $\lim_{n\to+\infty} u(t_0,x_n,y_n) - p^-(x_n,y_n) = 0$. Since $\phi > p^-$ and $\varepsilon_1 \in (0,1/2]$, it follows from (4.9) that $2\,\varepsilon_1\, g(s_{-\sigma_n}(t_0,x_n),x_n,y_n)\, e^{-\omega t_0} \ge -(\phi(s_{-\sigma_n}(t_0,x_n),x_n,y_n) - p^-(x_n,y_n))/2$, whence

$$\frac{\phi(s_{-\sigma_n}(t_0,x_n),x_n,y_n) - p^-(x_n,y_n)}{2\,(U(t_0,x_n,y_n) - p^-(x_n,y_n))} < \frac{u(t_0,x_n,y_n) - p^-(x_n,y_n)}{U(t_0,x_n,y_n) - p^-(x_n,y_n)}.$$

One gets a contradiction as in Step 2 of the proof of Proposition 3.2.

As a consequence, (4.15) holds for σ large enough and one can choose $\sigma_0 \ge \underline{\sigma}$ large enough so that (4.14) and (4.15) are fulfilled for $\sigma = \sigma_0$.

Let us now check that $\underline{u}$ and $\overline{u}$ are respectively sub- and super-solutions of (1.1) for $t \ge t_0$, when $\underline{u} > p^-$ and $\overline{u} < p^+$. Notice first that $\nu A(x,y)\nabla\underline{u}(t,x,y) = \nu A(x,y)\nabla\overline{u}(t,x,y) = 0$ as soon as $(x,y) \in \partial\Omega$, from (3.4), (4.2) and the definition of g. It follows from (4.10) and the definition of $s_{\sigma_0}(t,x)$ that

$$\begin{aligned}\mathcal{L}\underline{u}(t,x,y) \;=\;& f(x,y,\phi(s_{\sigma_0}(t,x),x,y)) - f(x,y,\underline{u}(t,x,y)) + \sigma_0\,\omega\,\phi_s(s_{\sigma_0}(t,x),x,y)\,e^{-\omega t}\\ & -2\,\varepsilon_1\, B(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-\omega t} - 2\,\varepsilon_1\,\sigma_0\,\omega\, C(s_{\sigma_0}(t,x)+s_0,x,y)\,e^{-2\omega t}.\end{aligned}$$

If $\underline{u}(t,x,y) > p^-(x,y)$ and $s_{\sigma_0}(t,x) \ge s^+$, where s^+ is given by (4.11), then it follows from the first four properties of (4.11) and from the inequalities $0 < \varepsilon_1 \le 1/2$ and $\phi_s < 0$, that $\mathcal{L}\underline{u}(t,x,y) \le 0$. If $\underline{u}(t,x,y) > p^-(x,y)$ and $s_{\sigma_0}(t,x) \le s^-$, where s^- is given by (3.15), then it follows from (3.15), $\phi_s < 0$, $s_0 \le 0$ and $\omega = \mu^+/2$ that $\mathcal{L}\underline{u}(t,x,y) \le 0$. Lastly, if $p^-(x,y) < \underline{u}(t,x,y)$ and $s^- \le s_{\sigma_0}(t,x) \le s^+$, then it follows from (3.16), (4.12), (4.13) and $\sigma_0 \ge \underline{\sigma}$ that $\mathcal{L}\underline{u}(t,x,y) \le 0$ too. Since $\underline{u}(t_0,x,y) \le u(t_0,x,y)$ and $p^-(x,y) \le u(t,x,y)$ for all $(t,x,y) \in [0,+\infty)\times\overline{\Omega}$, one concludes from the parabolic maximum principle that

$$\max\big[\underline{u}(t,x,y), p^-(x,y)\big] \le u(t,x,y) \;\text{ for all } (t,x,y) \in [t_0,+\infty)\times\overline{\Omega}.$$

Similarly, there holds

$$\begin{aligned}\mathcal{L}\overline{u}(t,x,y) \;=\;& f(x,y,\phi(s_{-\sigma_0}(t,x),x,y)) - f(x,y,\overline{u}(t,x,y)) + \sigma_0\,\omega\,\phi_s(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t}\\ & +2\,\varepsilon_1\, B(s_{-\sigma_0}(t,x),x,y)\,e^{-\omega t} - 2\,\varepsilon_1\,\sigma_0\,\omega\, C(s_{-\sigma_0}(t,x),x,y)\,e^{-2\omega t}.\end{aligned}$$

As above, it follows then from (3.15), (3.16), (4.11), (4.12), (4.13), and from $\phi_s < 0$, $s_0 \le 0$, $\omega = \mu^+/2$ and $\sigma_0 \ge \underline{\sigma}$ that $\mathcal{L}\overline{u}(t,x,y) \ge 0$ for all $(t,x,y) \in [t_0,+\infty) \times \overline{\Omega}$ such that $\overline{u}(t,x,y) < p^+(x,y)$. Since $\overline{u}(t_0,x,y) \ge u(t_0,x,y)$ and $u(t,x,y) \le p^+(x,y)$ for all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$, one concludes from the parabolic maximum principle that

$$u(t,x,y) \le \min\left[\overline{u}(t,x,y), p^+(x,y)\right] \quad \text{for all } (t,x,y) \in [t_0,+\infty) \times \overline{\Omega}.$$

Step 3: Time-global sharp estimates as $x \cdot e - c^*(e)t$ is very large. We now claim that, for any $\eta > 0$, there are $D_\eta > 0$ and $\sigma_\eta \in \mathbb{R}$ such that

$$\begin{array}{l}\forall\ (t,x,y) \in [0,+\infty) \times \overline{\Omega}, \quad [x \cdot e - c^*(e)t \ge \sigma_\eta] \Longrightarrow \\ \qquad \big[\phi(x \cdot e - c^*(e)t + \eta, x, y) - D_\eta\, \psi_{\lambda^*}(x,y)\, e^{-\lambda^*(x\cdot e - c^*(e)t)} \\ \qquad\qquad \le u(t,x,y) \le \phi(x \cdot e - c^*(e)t - \eta, x, y) + D_\eta\, \psi_{\lambda^*}(x,y)\, e^{-\lambda^*(x\cdot e - c^*(e)t)}\big].\end{array} \tag{4.16}$$

Let $\eta > 0$ be any positive real number. We are going to trap u, for very large $x \cdot e - c^*(e)t$, between a sub- and a super-solution which are larger and smaller than the left- and right-hand sides of (4.16), respectively.

Define, for all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$,

$$\underline{u}_\eta(t,x,y) = i(s+\eta/2,x,y) + h(s,x,y) - D_\eta\, \psi_{\lambda^*}(x,y)\, e^{-\lambda^* s} + p^-(x,y),$$

where $s = x \cdot e - c^*(e)t$, i and h have been defined in Step 1, and the real number $D_\eta > 0$ shall be chosen later. Remember now that the function $(x,y,\xi) \mapsto \frac{\partial f}{\partial u}(x,y,p^-(x,y)+\xi)$ is assumed to be of class $C^{0,\beta}(\overline{\Omega} \times [0,\gamma])$ for some $\beta > 0$ and $\gamma > 0$. Therefore, there exists a real number $r \ge 0$ such that

$$\left|f(x,y,p^-(x,y)+\xi) - f(x,y,p^-(x,y)) - \zeta^-(x,y)\,\xi\right| \le r\,\xi^{1+\beta} \tag{4.17}$$

for all $(x,y,\xi) \in \overline{\Omega} \times [0,\gamma]$. From (1.24) with $B = B_\phi$, (4.3), (4.5) and (4.8), there exists $\sigma_\eta \ge 0$ such that

$$\left\{\begin{array}{l} 0 < \phi(s+\eta,x,y) - p^-(x,y) \le i(s+\eta/2,x,y) + h(s,x,y) \le \gamma, \\ 0 < h(s,x,y) \le i(s+\eta/2,x,y), \\ r\,2^{1+\beta}\, i(s+\eta/2,x,y)^{1+\beta} \le \upsilon\ e^{-\mu s}\,(s+a)^{2m+2} \end{array}\right. \tag{4.18}$$

for all $(s,x,y) \in [\sigma_\eta,+\infty) \times \overline{\Omega}$, where $\upsilon > 0$ is given in (4.7), and

$$i(x \cdot e + \eta/2, x, y) + h(x \cdot e, x, y) \le u_0(x,y) - p^-(x,y) \tag{4.19}$$

for all $(x,y) \in \overline{\Omega}$ such that $x \cdot e \ge \sigma_\eta$. Then choose $D_\eta > 0$ large enough so that

$$i(\sigma_\eta + \eta/2, x, y) + h(\sigma_\eta, x, y) - D_\eta\, \psi_{\lambda^*}(x,y)\, e^{-\lambda^* \sigma_\eta} \le 0 \tag{4.20}$$

for all $(x,y) \in \overline{\Omega}$. In order to prove the first inequality of (4.16), it is then enough to prove, from (4.18), that

$$\underline{u}_\eta(t,x,y) \le u(t,x,y) \ \text{ for all } (t,x,y) \in [0,+\infty) \times \overline{\Omega} \ \text{ such that } x \cdot e - c^*(e)t \ge \sigma_\eta. \tag{4.21}$$

Observe that

$$\underline{u}_\eta(0,x,y) \le u_0(x,y) \ \text{ for all } (x,y)\in\overline{\Omega} \text{ such that } x\cdot e \ge \sigma_\eta$$

from (4.19), and that

$$\underline{u}_\eta(t,x,y) \le p^-(x,y) \le u(t,x,y) \text{ for all } (t,x,y)\in[0,+\infty)\times\overline{\Omega} \text{ such that } x\cdot e - c^*(e)t = \sigma_\eta,$$

from (4.20). Moreover, $\nu A(x,y)\nabla \underline{u}_\eta(t,x,y) = 0$ for all $(t,x,y)\in[0,+\infty)\times\partial\Omega$, since $\nu A(x,y)\nabla I = \nu A(x,y)\nabla H = \nu A(x,y)\nabla p^- = \nu A(x,y)\nabla\psi_{\lambda^*} - \lambda^*\,(\nu A(x,y)e)\,\psi_{\lambda^*} = 0$ for all $(x,y)\in\partial\Omega$. Lastly, remember that $u \ge p^-$. Therefore, from the parabolic maximum principle, in order to prove (4.21), it is enough to check that $\mathcal{L}\underline{u}_\eta(t,x,y)\le 0$ for all $(t,x,y)\in\Omega^-_\eta$, where

$$\Omega^-_\eta = \{(t,x,y)\in[0,+\infty)\times\overline{\Omega} \text{ such that } x\cdot e - c^*(e)t \ge \sigma_\eta \text{ and } \underline{u}_\eta(t,x,y) > p^-(x,y)\}.$$

From (4.4), (4.6) and $L_{\lambda^*}\psi_{\lambda^*} = k(\lambda^*)\,\psi_{\lambda^*}$ in $\overline{\Omega}$, it is straightforward to see that

$$\begin{array}{rcl}\mathcal{L}\underline{u}_\eta(t,x,y) & \le & \zeta^-(x,y)\,\big(i(s+\eta/2,x,y) + h(s,x,y) - D_\eta\,\psi_{\lambda^*}(x,y)\,e^{-\lambda^* s}\big)\\ & & -\upsilon\, e^{-\mu s}\,(s+a)^{2m+2} + f(x,y,p^-(x,y)) - f(x,y,\underline{u}_\eta(t,x,y))\end{array}$$

for all $(t,x,y)\in\Omega^-_\eta$, where $s = x\cdot e - c^*(e)t$. From (4.18), there holds $0 < \underline{u}_\eta(t,x,y) - p^-(x,y) \le \gamma$ for all $(t,x,y)\in\Omega^-_\eta$, whence

$$\begin{array}{rcl} f(x,y,\underline{u}_\eta(t,x,y)) & \ge & f(x,y,p^-(x,y)) + \zeta^-(x,y)\,(\underline{u}_\eta(t,x,y) - p^-(x,y))\\ & & -r\,(\underline{u}_\eta(t,x,y) - p^-(x,y))^{1+\beta}.\end{array}$$

Furthermore, $0 < \underline{u}_\eta(t,x,y) - p^-(x,y) \le 2\,i(x\cdot e - c^*(e)t + \eta/2,x,y)$ in Ω^-_η. Therefore, it follows that, for all $(t,x,y)\in\Omega^-_\eta$,

$$\mathcal{L}\underline{u}_\eta(t,x,y) \le -\upsilon\, e^{-\mu s}\,(s+a)^{2m+2} + r\,2^{1+\beta}\,i(s+\eta/2,x,y)^{1+\beta} \le 0$$

from (4.18). As a consequence, (4.21) holds, and then the first inequality in (4.16) as well.

Define now, for all $(t,x,y)\in[0,+\infty)\times\overline{\Omega}$,

$$\overline{u}_\eta(t,x,y) = i(s-\eta/2,x,y) + D_\eta\,\psi_{\lambda^*}(x,y)\,e^{-\lambda^* s} + p^-(x,y),$$

where $s = x\cdot e - c^*(e)t$. Even if it means increasing σ_η and D_η, it follows from (1.24) with $B = B_\phi$, (4.3) and (4.8) that one can assume that (4.18), (4.19) and (4.20) hold, as well as

$$\left\{\begin{array}{l} \forall\ (s,x,y)\in[\sigma_\eta,+\infty)\times\overline{\Omega},\quad 0 < i(s-\eta/2,x,y) \le \phi(s-\eta,x,y) - p^-(x,y),\\ \forall\ (x,y)\in\overline{\Omega},\quad [x\cdot e \ge \sigma_\eta] \Longrightarrow [u_0(x,y) - p^-(x,y) \le i(x\cdot e - \eta/2,x,y)],\\ \forall\ (x,y)\in\overline{\Omega},\quad i(\sigma_\eta - \eta/2,x,y) + D_\eta\,\psi_{\lambda^*}(x,y)\,e^{-\lambda^*\sigma_\eta} + p^-(x,y) \ge p^+(x,y).\end{array}\right. \tag{4.22}$$

In order to prove the second inequality of (4.16), it is then enough to prove that

$$u(t,x,y) \le \overline{u}_\eta(t,x,y) \ \text{ for all } (t,x,y)\in[0,+\infty)\times\overline{\Omega} \ \text{ such that } x\cdot e - c^*(e)t \ge \sigma_\eta. \tag{4.23}$$

It also follows from (4.22) that

$$u_0(x,y) \leq \overline{u}_\eta(0,x,y) \text{ for all } (x,y) \in \overline{\Omega} \text{ such that } x\cdot e \geq \sigma_\eta,$$

and that

$$u(t,x,y) \leq p^+(x,y) \leq \overline{u}_\eta(t,x,y) \text{ for all } (t,x,y) \in [0,+\infty)\times\overline{\Omega} \text{ such that } x\cdot e - c^*(e)t = \sigma_\eta.$$

Moreover, $\nu A(x,y)\nabla\overline{u}_\eta(t,x,y) = 0$ for all $(t,x,y) \in [0,+\infty)\times\partial\Omega$. Remember that $u \leq p^+$. Therefore, from the parabolic maximum principle, in order to prove (4.23), it is enough to check that $\mathcal{L}\overline{u}_\eta(t,x,y) \geq 0$ for all $(t,x,y) \in \Omega_\eta^+$, where

$$\Omega_\eta^+ = \{(t,x,y) \in [0,+\infty)\times\overline{\Omega} \text{ such that } x\cdot e - c^*(e)t \geq \sigma_\eta \text{ and } \overline{u}_\eta(t,x,y) < p^+(x,y)\}.$$

From (4.4), from $L_{\lambda^*}\psi_{\lambda^*} = k(\lambda^*)\,\psi_{\lambda^*}$ and from the KPP condition (1.6), there holds

$$\begin{array}{rcl}\mathcal{L}\overline{u}_\eta(t,x,y) & = & \zeta^-(x,y)\left(i(s-\eta/2,x,y) + D_\eta\,\psi_{\lambda^*}(x,y)\,e^{-\lambda^* s}\right)\\ & & +f(x,y,p^-(x,y)) - f(x,y,\underline{u}_\eta(t,x,y))\\ & \geq & 0\end{array}$$

for all $(t,x,y) \in \Omega_\eta^+$, where $s = x\cdot e - c^*(e)t$. As a consequence, (4.23) holds, and then the second inequality in (4.16) as well.

Step 4: Conclusion. By using the fact that $e^{-\lambda^* s} = o(\phi(s,x,y) - p^-(x,y))$ as $s \to +\infty$ uniformly in $(x,y) \in \overline{\Omega}$, it follows from the same arguments as in Proposition 3.4 that, if $v(t,x,y)$ is a solution of (1.1) in $\mathbb{R}\times\overline{\Omega}$ such that

$$\left\{\begin{array}{l}\exists\, a \geq b \in \mathbb{R},\quad \phi(x\cdot e - c^*(e)t + a,x,y) \leq v(t,x,y) \leq \phi(x\cdot e - c^*(e)t + b,x,y) \text{ in } \mathbb{R}\times\overline{\Omega},\\ \forall\,\eta > 0,\ \exists\, D_\eta > 0,\ \exists\,\sigma_\eta \in \mathbb{R},\quad [s = x\cdot e - c^*(e)t \geq \sigma_\eta] \Longrightarrow\\ \qquad \left[\phi(s+\eta,x,y) - D_\eta\,\psi_{\lambda^*}(x,y)\,e^{-\lambda^* s} \leq v(t,x,y) \leq \phi(s-\eta,x,y) + D_\eta\,\psi_{-\lambda^*}(x,y)\,e^{-\lambda^* s}\right],\end{array}\right.$$

then $v(t,x,y) = \phi(s,x,y) = U(t,x,y)$ in $\mathbb{R}\times\overline{\Omega}$.

Finally, from this Liouville type result and Steps 2 and 3, the proof of property (1.25) of part 2) of Theorem 1.5 (with $\tau = 0$ due to our assumption $B = B_\phi$) can be done with the same arguments as those used in the proof of property (1.21) of Theorem 1.3. The proof of Theorem 1.5 is then complete. □

References

[1] Bages, M., Martinez, P., Roquejoffre, J.-M.: Dynamique en temps grand pour une classe d'équations de type KPP en milieu périodique, C. R. Acad. Sci. Paris Ser. I **346** (2008) 1051-1056.

[2] Berestycki, H., Hamel, F.: Front propagation in periodic excitable media, Comm. Pure Appl. Math. **55** (2002) 949-1032.

[3] Berestycki, H., Hamel, F.: Reaction-Diffusion Equations and Propagation Phenomena, Applied Mathematical Sciences, Springer-Verlag, to appear.

[4] Berestycki, H., Hamel, F., Nadirashvili, N.: The principal eigenvalue of elliptic operators with large drift and applications to nonlinear propagation phenomena, Comm. Math. Phys. **253** (2005) 451-480.

[5] Berestycki, H., Hamel, F., Nadirashvili, N.: The speed of propagation for KPP type problems. I - Periodic framework, J. Europ. Math. Soc. **7** (2005) 173-213.

[6] Berestycki, H., Hamel, F., Roques, L.: Analysis of the periodically fragmented environment model : I - Species persistence, J. Math. Biol. **51** (2005) 75-113.

[7] Berestycki, H., Hamel, F., Roques, L.: Analysis of the periodically fragmented environment model : II - Biological invasions and pulsating traveling fronts, J. Math. Pures Appl. **84** (2005) 1101-1146.

[8] Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method, Bol. Soc. Bras. Mat. **22** (1991) 1-37.

[9] Berestycki, H., Nirenberg, L.: Travelling fronts in cylinders, Ann. Inst. H. Poincaré, Anal. Non Lin. **9** (1992) 497-572.

[10] Bramson, M.: Convergence of solutions of the Kolmogorov equation to traveling waves, Memoirs Amer. Math. Soc. **44**, 1983.

[11] Cabré, X., Roquejoffre, J.-M.: Propagation de fronts dans les équations de Fisher-KPP avec diffusion fractionnaire, Preprint.

[12] Conca, C., Vanninathan, M.: Homogenization of periodic structures via Bloch decomposition, SIAM J. Appl. Math. **57** (1997) 1639-1659.

[13] El Smaily, M.: Pulsating traveling fronts: Asymptotics and homogenization regimes, Europ. J. Appl. Math. **19** (2008) 393-434.

[14] El Smaily, M., Hamel, F., Roques, L.: Homogenization and influence of fragmentation in a biological invasion model, Disc. Cont. Dyn. Syst. A **25** (2009) 321-342.

[15] Fife, P.C., McLeod, J.B.: The approach of solutions of non-linear diffusion equations to traveling front solutions, Arch. Ration. Mech. Anal. **65** (1977) 335-361.

[16] Fisher, R.A.: The advance of advantageous genes, Ann. Eugenics **7** (1937) 335-369.

[17] Fréjacques, G.: Travelling waves in infinite cylinders with time-periodic coefficients, Ph. D. Thesis, 2005.

[18] Gallay, T.: Local stability of critical fronts in nonlinear parabolic pde's, Nonlinearity **7** (1994) 741-764.

[19] Guo, J.-S., Wu, C.-C.: Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system, Preprint.

[20] Hamel, F.: Qualitative properties of monostable pulsating fronts : exponential decay and monotonicity, J. Math. Pures Appl. **89** (2008) 355-399.

[21] Hamel, F., Nadin, G.: Large-time KPP spreading speeds with exponentially oscillating initial data, Preprint.

[22] Hamel, F., Roques, L.: Fast propagation in reaction-diffusion equations with slowly decaying initial conditions, Preprint.

[23] Hamel, F., Sire, Y.: Spreading speeds for some reaction-diffusion equations with general initial conditions, Preprint.

[24] Heinze, S.: Large convection limits for KPP fronts, Preprint.

[25] Hudson, W., Zinner, B.: Existence of traveling waves for reaction-diffusion equations of Fisher type in periodic media, In : Boundary Problems for Functional Differential Equations, World Scientific, 1995, 187-199.

[26] Kametaka, Y.: On the nonlinear diffusion equation of Kolmogorov-Petrovski-Piskunov type, Osaka J. Math. **13** (1976) 11-66.

[27] Kanel', Ya.I.: Stabilization of solution of the Cauchy problem for equations encountred in combustion theory, Mat. Sbornik **59** (1962) 245-288.

[28] Kato, T.: Perturbation theory for linear operators, 2nd edition, Springer, 1984.

[29] Kolmogorov, A.N., Petrovsky, I.G., Piskunov, N.S.: Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bull. Univ. Etat Moscou (Bjul. Moskowskogo Gos. Univ.), Sér. intern. **A 1** (1937) 1-26.

[30] Larson, D.A.: Transient bounds and time-asymptotic behavior of solutions to nonlinear equations of Fisher type, SIAM J. Appl. Math. **34** (1978) 93-103.

[31] Lau, K.-S.: On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov, J. Diff. Equations **59** (1985) 44-70.

[32] Levermore, C.D., Xin, J.X.: Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, II, Comm. Part. Diff. Equations **17** (1999) 1901-1924.

[33] Mallordy, J.-F., Roquejoffre, J.-M.: A parabolic equation of the KPP type in higher dimensions, SIAM J. Math. Anal. **26** (1995) 1-20.

[34] Matano, H.: Traveling waves in spatially inhomogeneous diffusive media – The non-periodic case, Preprint.

[35] Murray, J.D.: Mathematical biology, 3rd edition in 2 volumes: I. An Introduction, Springer, 2002 (corr. 2nd printing 2008); II. Spatial models and biomedical applications, Springer, 2003.

[36] Nadin, G.: Travelling fronts in space-time periodic media, J. Math. Pures Appl., to appear.

[37] Nolen, J., Xin, J.: Reaction-diffusion front speeds in spatially-temporally periodic shear flows, SIAM Multi. Model. Simul. **1** (2003) 554-570.

[38] Nolen, J., Xin, J.: Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle, Disc. Cont. Dyn. Syst. **13** (2005) 1217-1234.

[39] Novikov, A., Ryzhik, L.: Boundary layers and KPP fronts in a cellular flow, Arch. Ration. Mech. Anal. **184** (2007) 23-48.

[40] Roquejoffre, J.-M.: Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders, Ann. Inst. H. Poincaré, Anal. Non Lin. **14** (1997) 499-552.

[41] Ryzhik, L., Zlatos, A.: KPP pulsating front speed-up by flows, Commun. Math. Sci. **5** (2007) 575-593.

[42] Sattinger, D.H.: Stability of waves of nonlinear parabolic systems, Adv. Math. **22** (1976) 312-355.

[43] Shigesada, N., Kawasaki, K.: Biological Invasions: Theory and Practice, Oxford Series in Ecology and Evolution, Oxford : Oxford UP, 1997.

[44] Uchiyama, K.: The behavior of solutions of some semilinear diffusion equation for large time, J. Math. Kyoto Univ. **18** (1978) 453-508.

[45] van Saarloos, W.: Front propagation into unstable states, Physics Reports **386** (2003) 29-222.

[46] Weinberger, H.F.: On spreading speeds and traveling waves for growth and migration in periodic habitat, J. Math. Biol. **45** (2002) 511-548.

[47] Xin, X.: Existence and stability of traveling waves in periodic media governed by a bistable nonlinearity, J. Dyn. Diff. Equations **3** (1991) 541-573.

[48] Xin, X.: Existence of planar flame fronts in convective-diffusive periodic media, Arch. Ration. Mech. Anal. **121** (1992) 205-233.

[49] Xin, J.X.: Analysis and modeling of front propagation in heterogeneous media, SIAM Review **42** (2000) 161-230.

[50] Zlatos, A.: Sharp asymptotics for KPP pulsating front speed-up and diffusion enhancement by flows, Arch. Ration. Mech. Anal., to appear.